

## ADVANCED MATHEMATICS IN ECONOMICS

### PROBLEM SET 1

Due by Oct 4 (before the tutorial)

#### Problem 1 - Deductive Proofs (4 Points)

(a) Show that the following statement is true:

$$\forall a, b \in \mathbb{R} \text{ with } a < b: a < \frac{a+b}{2} < b.$$

(b) Let  $A \subset \mathbb{R}$  be some finite subset of  $\mathbb{R}$ , i.e.  $A = \{a_1, a_2, \dots, a_n\}$ ,  $n \in \mathbb{N}$ . Furthermore, without loss of generality let  $\min\{a_1, \dots, a_n\} = a_1$  and  $\max\{a_1, \dots, a_n\} = a_n$ . Finally, let  $\{p_1, p_2, \dots, p_n; p_i \in \mathbb{R}^+\}$  be a set of weight factors. Show that

$$\min\{a_1, \dots, a_n\} \leq \frac{\sum_{i=1}^n p_i a_i}{\sum_{i=1}^n p_i} \leq \max\{a_1, \dots, a_n\}$$

HINT: By definition, we see that  $a_1 \leq a_k \leq a_n \forall k = 1, \dots, n$ . In other words, we have a system of  $n$  inequalities. Now, for  $k = 1, \dots, n$ , you need to multiply the  $k$ -th inequality with weight factor  $p_k$ . Then you need to sum all  $n$  weighted inequalities. You will get another inequality that looks very similar to the one from the statement. Some minor manipulations and the fact that all transformations were equivalence transformations yield the claim.

#### Problem 2 - Induction Proofs (4 Points)

(a) *Geometric sum*: Prove via induction:

$$\forall x \in \mathbb{R}, n \in \mathbb{N}: (1-x) \sum_{k=0}^n x^k = 1 - x^{n+1}$$

NOTE: Together with the concept of a limit in  $\mathbb{R}$ , this result directly implies that the geometric series converges to  $1/(1-x) \forall x \in [0; 1)$ .

(b) *Bernoulli's inequality*- Prove via induction:

$$\forall n \in \mathbb{N}, x \in \mathbb{R}, x \geq -1: (1+x)^n \geq 1 + nx.$$

Highlight why we require  $x \geq -1$ .

**Problem 3 - Proofs by Contradiction (5 Points)**

- (a) Prove De Morgan's second law: Let  $A, B \subset U$  be two arbitrary sets where  $U$  denotes the universal set. Formally show that

$$(A \cap B)^c = A^c \cup B^c.$$

HINT: You need to show that  $(a \in (A \cap B)^c) \Leftrightarrow (a \in (A^c \cup B^c))$ . To prove equivalence, you have to split the proof into a part where you show first direction " $P \Rightarrow Q$ " and " $P \Leftarrow Q$ ". For each direction you have to show the contradiction, e.g. for " $P \Rightarrow Q$ ", assume that  $Q$  is not true and show that this implies  $P$  is not true.

- (b) Let  $A := \{r \in \mathbb{R} : r = (-1)^k \frac{1}{k} + 1, k \in \mathbb{N} \setminus \{0\}\}$ . Show that  $A \subseteq [0; \frac{3}{2}]$ .

HINT: You need to discuss separately the elements of  $A$  generated by odd  $k$  and the elements of  $A$  generated by even  $k$ .

**Problem 4 - Surjectivity, Injectivity, Bijectivity (2 Points)**

Let  $f : X \rightarrow Y$  be a mapping and let  $id_X : X \rightarrow X$ , with  $id_X(x) = x$  be the identity on  $X$ .

- (a) Show that  $f$  is injective  $\Leftrightarrow \exists \tilde{f} : Y \rightarrow X$  with  $\tilde{f} \circ f = id_X$ .

- (b) Show that  $f$  is surjective  $\Leftrightarrow \exists \bar{f} : Y \rightarrow X$  with  $f \circ \bar{f} = id_Y$ .

NOTE: Both results together directly imply that for  $f$  bijective, there is a unique  $f^{-1} : Y \rightarrow X$  with  $f^{-1} \circ f = id_X \wedge f \circ f^{-1} = id_Y$ . This  $f^{-1}$  is called the inverse of  $f$ .