ADVANCED MATHEMATICS IN ECONOMICS
PROBLEM SET 2

Due by Oct 8 (before the tutorial)

Problem 1 - Limits of Sequences (8 Points)

(a) Calculate the limit of $a_n = \frac{n^2+n+2}{4n^3+1}$ as $n \to \infty$.

(b) Calculate the limit of $b_n = \frac{n^3}{\binom{2n}{n}}$, where $\binom{n}{k} := \frac{n!}{(n-k)!k!}$ and $n! := \prod_{l=1}^{n} l$.

(c) Calculate the limit of $c_n = \sqrt{n}(\sqrt{n+1} - \sqrt{n})$.
HINT: Expand c_n by multiplying with $1 = \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$. Then cancel out $\sqrt{n}$ in nominator and denominator.

(d) Find the limit of $d_n = \sum_{i=0}^{n} a^i$ with $0 < a < 1$.

(e) Find the limit of $e_n = \sum_{i=1}^{n} \frac{1}{i}$.

(f) Consider the metric space $(\mathbb{R}, |\cdot|)$. Prove or disprove the following statement: $(a_n)_{n\in\mathbb{N}}$ converges $\Leftrightarrow$ $(|a_n|)_{n\in\mathbb{N}}$ converges.
HINT: Use the reverse triangle inequality: $||x| - |y|| \leq |x - y| \; \forall x, y \in \mathbb{R}$.

Problem 2 - Topology (4 Points)

(a) Prove or disprove: $d_1(x, y) := e^{x-y}$ is a metric in $\mathbb{R}$.

(b) Prove or disprove:

$$d_2(x, y) := \begin{cases} 0, \; x = y \\ 1, \; x \neq y \end{cases}$$

is a metric in $\mathbb{R}^n$. If you find that this is in fact a metric, discuss whether it is induced by a norm. Prove your claim.

(c) Consider the metric space $(\mathbb{R}^n, d)$. Show that $B_1(x) = \{y \in \mathbb{R}^n : d(x, y) < 1\}$ is open in $\mathbb{R}^n$.

Problem 3 - Cauchy-Sequences (2 Points)

Consider the metric space $(\mathbb{R}, |\cdot|)$. Let $a_{n+1} := 1 + \frac{1}{1+a_n}$ and $a_1 = 1$. Show that (a_n) is a Cauchy-Sequence.
HINTS:

- In a first step, you need to prove that

$$|a_m - a_n| \leq \left(\frac{1}{2}\right)^{n-1} \quad \forall m > n.$$

  For this, you have to set $m = n + k$ and perform an induction proof over $k \in \mathbb{N}$.

- In a second step you need to use this result to argue that (a_n) fulfills the definition of a Cauchy-Sequence.

Problem 4 - Continuity of Functions (6 Points)

(a) Let $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} e^{-1/x^2}, \ x \neq 0 \\ 0, \ x = 0 \end{cases}$$

Show that $f(x)$ is continuous $\forall x \in \mathbb{R}$ using the limit definition of continuity.

(b) Show that $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $f(x) = \sqrt{x}$ is continuous $\forall x \in \mathbb{R}_0^+$ by using the epsilon-delta definition of continuity.
HINT: You may use the fact that $\sqrt{x}$ is monotonously increasing on $\mathbb{R}_0^+$.

(c) Let

$$g(x) := \begin{cases} \sin\left(\frac{1}{x}\right), \ x \neq 0 \\ 0 \ , \ x = 0 \end{cases}$$

and let

$$h(x) := \begin{cases} x \sin\left(\frac{1}{x}\right), \ x \neq 0 \\ 0 \ , \ x = 0 \end{cases}$$

Prove or disprove that g, h are continuous in 0 by using the sequence definition of a function limit.

(d) Now consider $a_n := sin(1/n)$, $\forall n \in \mathbb{N}$. Does this sequence have a convergent subsequence? Why or why not? How many different accumulation points does this sequence have? What are they?