ADVANCED MATHEMATICS IN ECONOMICS
PROBLEM SET 3

Due by Oct 9

Problem 1 - Matrix Multiplication (3 Points)

Prove or disprove the following statements:

(a) Let $A, B \in \mathbb{R}^{n\times n}$. Then $A \cdot B = B \cdot A$ holds.

(b) Let $A, B \in \mathbb{R}^{n\times n}$ be upper triangular matrices. Then $A \cdot B = C$ is again an upper triangular matrix.
HINT: $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ is upper triangular iff $a_{ij} = 0 \forall i > j$. You may also use the fact that $\sum_{k=m}^{n} a_k := 0 \, \forall m > n$.

Problem 2 - Span, Basis and Linear Independence (2 Points)

Which of the following are a basis in $\mathbb{R}^3$? Explain.

(1) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$; (2) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$; (3) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; (4) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Problem 3 - Trace, Determinant, Rank, Inverse of a Matrix (8 Points)

(a) Let $D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.
Calculate the trace and the rank of this matrix. What can be inferred regarding the determinant? What is D^{-1} in this case? Justify your claim.

(b) Calculate the determinant of $C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 2 \end{pmatrix}$.

Show that $C^{-1} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$.

(c) Prove the cyclic property of the trace, i.e.
Let $A, B, C, D \in \mathbb{R}^{n \times n}$. Then:

$$tr(ABCD) = tr(BCDA) = tr(CDAB) = tr(DABC)$$

HINT: Matrix multiplication is associative.

(d) Let $A, B \in \mathbb{R}^{n \times n}$, where A is singular. What does this imply for

(1) $(AB)^{-1}$?

(2) $rank(AB)$?

Problem 4 - Idempotency (4 Points)

(a) Prove or disprove the following statement: All idempotent matrices are symmetric.

(b) Prove or disprove the following statement: Let $X \in \mathbb{R}^{n \times k}$. Then $X(X'X)^{-1}X'$ is idempotent and symmetric.

(c) Consider the estimated linear system $y = X\hat{\beta} + e$, where $\hat{\beta} := (X'X)^{-1}X'y$. Show that the following holds for the sum of squared residuals $e'e$:
$e'e = y'My$, where M is an idempotent matrix.

Problem 5 - Eigenvalues and Eigendecomposition (8 Points)

(a) Let $P = \begin{pmatrix} 0.8 & 0.2 \\ 0.05 & 0.95 \end{pmatrix}$.

(1) Calculate the eigenvalues and normalized eigenvectors of P.

(2) Use the eigendecomposition to calculate

$$\lim_{n \to \infty} P^n.$$

HINT: The limit values apply to matrix-values sequences as well.
E.g. Let $(A_n)_{n \in \mathbb{N}}$, $(B_n)_{n \in \mathbb{N}}$ be matrix-valued sequences. Then

$$\lim_{n \to \infty} (A_n \cdot B_n) = \lim_{n \to \infty} A_n \cdot \lim_{n \to \infty} B_n$$

Applying the limit to a matrix means to apply it to every component of the matrix.

(b) Let $A \in \mathbb{R}^{n \times n}$ with n distinct eigenvalues $\lambda_1, \cdots, \lambda_n$. Show that

$$tr(A) = \sum_{i=1}^{n} \lambda_i.$$

HINT: Use the eigen decomposition and Problem 2 (c).