ADVANCED MATHEMATICS IN ECONOMICS
PROBLEM SET 4

Due by Oct 11

Problem 1 - Taylor Series Approximation and Power Series (5 Points)

(a) Develop the Taylor series of $f(x) = \ln(\frac{1}{2}x)$ around $\bar{x} = 2$. For which $x \in \mathbb{R}$ does the Taylor series converge? How are these limit values related to $\ln(\frac{1}{2}x)$?
HINTS: (1) Let $a_n \in \mathbb{R}_{\neq 0}$ for $n \in \mathbb{Z}$. Then $\sum_{n=1}^{\infty} a_n < \infty$ if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$. On the other hand, if the latter is strictly greater than 1, the sum diverges.
(2) The alternating harmonic series equals $\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{n} = \ln(2)$.

(b) Use a Taylor approximation of order 1 around some steady state $\tilde{y}$ to log-linearize the following equations:

(b1) $y_t = (a x_t^{\theta} + 1)^{\frac{1}{\theta}}$

(b2) $y_t = x_t^{\alpha} z_t^{\beta}$

Problem 2 - One-dimensional Calculus (3 Points)

Let $f(x) = x^x$. Find all global and local extrema of f on $(0;3]$.
HINT: $x^x := e^{x\cdot\ln(x)}$

Problem 3 - Continuity in $\mathbb{R}^n$ - (4 Points)

Let $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

(a) **Optional from class:** Use the sequence definition of continuity to test whether f is continuous in $(0,0)$.

(b) Calculate the partial derivatives for f for $(x,y) \neq (0,0)$ and show that they also exist for $(x,y) = (0,0)$ by using the definition of partial derivatives.

(c) Are (a) and (b) together a contradiction to the fact differentiability implies continuity?

Problem 4 - Differentiation of Utility Functions (5 Points)

Calculate the gradient and the Hessian of

(a) $u(x_1, x_2) = [\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}]^{\frac{1}{\rho}}$ (CES)

(b) $u(x_1, x_2) = x_1^{\alpha_1} \cdot x_2^{\alpha_2}$ (Cobb-Douglas)

where $x_1, x_2 \in \mathbb{R}_0^+$, $\alpha_1, \alpha_2 \in \mathbb{R}^+$ and $\rho \geq 1$.

(c) Discuss what your findings in (a) and (b) imply for global and local extrema of utility functions.

Problem 5 - Multi-dimensional Chain Rule (3 Points - optional)

Let $g: \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \to (x + y, x - y)$ and $f: \mathbb{R}^2 \to \mathbb{R}$, $(u, v) \to \sin(u^2 - v^2)$.

(a) Calculate the Jacobian $Df(g(x, y))$ using the multi-dimensional chain rule.

(b) Suppose $g: \mathbb{R}^2 \to \mathbb{R}$ with $(x, y) \to x + y$ while f remains the same. Is the composition $f(g(x, y))$ still well-defined? Justify.

Problem 6 - Integration (5 Points)

(a) Calculate $\int_0^1 \frac{2x\, ln(x^2+1)}{x^2+1}\, dx$.

(b) Calculate $\int_0^1 x^2 e^{2x} dx$ using integration by parts.

(c) Calculate $\int_1^2 \frac{3x+6}{x^2+4x-4} dx$ using integration by substitution.

(d) Let $f: [0;1] \to \mathbb{R}$ be a two times continuously differentiable function with $f(0) = 1$, $f(1) = 3$ and $f'(1) = 0$. Calculate $\int_0^1 2x f''(x)\, dx$.

Problem 7 - Constrained Optimisation (6 Points)

(a) Utility is given by $u(a, b) = a^{\alpha} b^{\beta}$ and the available budget is B. Let p_a and p_b be prices and assume $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ and $\alpha + \beta = 1$. Maximise utility subject to the budget constraint.

(b) Characterise the solution found in (a). Is it global or local?

An agent lives in continuous time from $t = 0$ to $t = 1$ and is provided with an initial capital stock $k(0) = 1$. He or she wants to maximise utility $u(t) = k(t) + c(t)$ where $c(t)$ is consumption. The capital stock changes according to $dk(t)/dt = 1 - c(t)^2$. The agent would like to know which paths of capital and consumption would be optimal.

(c) What is the state and what is the control variable?

(d) Formulate the optimisation problem by setting up the Hamiltonian.

(e) What value will capital have in $T = 1$?

(f) Use your insight from (e), the constraints and Pontryagin's maximum principle to solve the problem in (d).