

ADVANCED MATHEMATICS IN ECONOMICS

PROBLEM SET 5

Due by Oct 11

Problem 1 - Probability Theory (9 Points)

(a) Empirical and theoretical means and variances:

Let x_1, \dots, x_n be realized observations of some random variable X , which is distributed according to some distribution P such that $E[X]$ and $E[X^2]$ exist.

Let $\hat{\mu}_X := \frac{1}{n} \sum_{i=1}^n x_i$ denote the empirical mean of x_1, \dots, x_n .

(i) Show that $\sum_{i=1}^n (x_i - \hat{\mu}_X) = 0$

(ii) Show that $\frac{1}{n} \sum_{i=1}^n (x_i - \hat{\mu}_X)^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 - \hat{\mu}_X^2$.

(iii) Do similar results hold for the theoretical distribution of X ? Justify your claim.

(b) The exponential distribution:

Let

$$f_\lambda(x) := \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Show that $f_\lambda(x)$ constitutes a probability density. Calculate $E[f_1(x)]$.

(c) Bayesian probability and basic concepts in probability theory:

Let $V \in \{0; 1\}$ with $P[V = 0] = p$, $P[V = 1] = 1 - p$. Further, let us assume we receive one of two noisy signals on the true state of V :

$P[S V]$	$V = 0$	$V = 1$
$S = 0$	0,8	0,4
$S = 1$	0,2	0,6

- (i) Do the rows and the columns of the $P[S|V]$ -matrix constitute a probability distribution?
- (ii) Calculate $E[V|S = 0]$ using Bayes' law. Compare your result to $E[V]$ and interpret. Hint: Bayes' Law is given by $P(V|S) = \frac{P(S|V)P(V)}{P(S)}$.
- (iii) Assume now that $p = 0.5$ and that another player revealed to you that he received a signal $S = 0$. Calculate the probability you (as a Bayesian) should assign to the state $V = 1$, given you received a signal $S = 1$. Interpret.

Problem 2 - Conditional Expectation (4 Points)

Let X, Y be two independent random variables describing the rolling of a six-sided dice. Let $Z := 2X + Y^2$.

Hint: Sometimes probabilities are tedious.

- (a) Calculate $E[Z]$.
- (b) Calculate $E[Z|X = 5]$ and $E[X|Z = 8]$.
- (c) Calculate $E[Z|X]$ and $E[X|Z]$. How do these objects differ from those in Problem 2(b)?
- (d) Use your result from 2(c) to calculate $E[E[Z|X]]$ and $E[E[X|Z]]$. Explain how your results relate to the law of iterated expectations.

Problem 3 - Transformations (2 Points)

Let $X \sim U[0, 2]$. What is the density function of $Y = \ln(X + 1)$? Use this example to show that Jensen's Inequality holds.