Deutsches Institut für
Wirtschaftsforschung

DIW BERLIN

Discussion Papers

930

Irina Georgescu

Arrow Index of a Fuzzy Choice Function

Berlin, October 2009

IMPRESSUM

DIW Berlin
German Institute for Economic Research
Mohrenstr. 58
10117 Berlin
Tel. +49 (30) 897 89-0
Fax +49 (30) 897 89-200
http://www.diw.de

ISSN print edition 1433-0210
ISSN electronic edition 1619-4535

Available for free downloading from the DIW Berlin website.

Discussion Papers of DIW Berlin are indexed in RePEc and SSRN.
Papers can be downloaded free of charge from the following websites:

http://www.diw.de/english/products/publications/discussion_papers/27539.html
http://ideas.repec.org/s/diw/diwwpp.html
http://papers.ssrn.com/sol3/JELJOUR_Results.cfm?form_name=journalbrowse&journal_id=1079991

Arrow index of a fuzzy choice function

Irina Georgescu*

Academy of Economic Studies,
Dept. of Economic Cybernetics,
Piata Romana No 6, Bucharest, Romania and
Deutsches Institut fur Wirtschaftsforschung,
Mohrenstrasse 58
10117 Berlin
Germany

Abstract

The Arrow index of a fuzzy choice function C is a measure of the degree to which C satisfies the Fuzzy Arrow Axiom, a fuzzy version of the classical Arrow Axiom. The main result of this paper shows that $\mathcal{A}(C)$ characterizes the degree to which C is full rational. We also obtain a method for computing $\mathcal{A}(C)$. The Arrow index allows to rank the fuzzy choice functions with respect to their rationality. Thus, if for solving a decision problem several fuzzy choice functions are proposed, by the Arrow index the most rational one will be chosen.

Keywords: fuzzy choice function, revealed preference indicator, congruence indicator, similarity

1 Introduction

One of the themes of classical economic theory is the rationality of the consumer behaviour. Revealed preference is a concept introduced by Samuelson in 1938 [26], aiming to postulate the rationality of a consumer's behaviour in terms of a preference relation associated with a demand function. Uzawa [35] and Arrow [1] have developed a revealed preference theory in an abstract framework. The work of Uzawa and Arrow was continued by Richter [24], Sen [27, 28, 29], Suzumura [30, 31] and many others.

Because of the insufficient information and human subjectivity the preferences of the individuals are often fuzzy. For this reason instead of saying that alternative x is better than alternative y, it is better to evaluate the degree of

*Correspondence address: P. O. Box 1–432, 014700, Bucharest, Romania *E-mail address*: crinaus2003@yahoo.com (I. Georgescu)

preference of x to y, this will be always a number in $[0, 1]$. The idea of mathematical modeling of vague preferences is obvious: given a set X of alternatives, the preferences will be represented by a binary fuzzy relation on X [8, 9, 10, 22].

Even if the preference is ambiguous, the choice can be exact or vague. Some authors [3, 4, 5, 20] study various crisp choice functions based on a fuzzy preference relation. There are cases (e. g. negotiations on electronic marketplaces) when the decision maker cannot make a definitive choice [17, 21, 33, 34]. In this process of decision making, the choice is potential [2]. Therefore, the act of choice will be modeled by a fuzzy choice function.

Fuzzy choice functions have been studied by several authors [2, 7, 10, 20, 22, 25, 34]. In particular, Banerjee developed in [2] a theory of revealed preference for a class of fuzzy choice functions. A Banerjee choice function has the domain consisting of crisp sets and the range consisting of fuzzy sets.

In [7] Dasgupta and Deb study properties of rationality for a class of fuzzy choice functions. The authors consider that:

> Even when one is analyzing *precise* choice when preferences are fuzzy, this approach is useful. Fuzzy choice sets provide an important intermediate step analogous to the "substitution effect" in the theory of consumer demand. Thus, precise choice with fuzzy preference may be viewed as taking place in two steps: (a) fuzzy choice; (b) fuzzy choice being "made" precise in some "natural" way.

In [11, 12, 13, 14, 15, 16] we worked with a more general definition of fuzzy choice functions. Banerjee fuzzifies only the range of a choice function, in our approach both the domain and the range of a choice function are made of fuzzy subsets of a universe of alternatives.

Works [12, 13, 14, 15, 16] are an attempt to build a theory of revealed preference for these fuzzy choice functions. In [15] we have introduced the Fuzzy Arrow Axiom (FAA) and we have proved that a fuzzy choice function satisfies (FAA) iff it is full rational. This result extends a classical result of [1] which establishes the equivalence between the Arrow Axiom and the full rationality of crisp choice functions (see also [31]). In this paper there is also defined the Arrow index $\mathcal{A}(C)$ of a fuzzy choice function C. $\mathcal{A}(C)$ is a number which represents the degree to which C satisfies the Fuzzy Arrow Axiom.

The main result of this paper is a characterization theorem for the Arrow index $\mathcal{A}(C)$. Intuitively, this theorem shows that $\mathcal{A}(C)$ characterizes the degree to which the fuzzy choice function C is full rational. At the same time, this theorem refines the mentioned result from [15] which is obtained as a particular case.

The main theorem of [14] established the equivalence between (FAA) and the full rationality of fuzzy choice functions. By measuring by the Arrow index the degree of full rationality of a fuzzy choice function, the results of this paper allow to compare them from the point of view of their rationality. Then by using the Arrow index one can obtain a hierarchy of fuzzy choice functions with respect to their full rationality.

2 Preliminaries

In this section we present some basic matter on the residuated structure of the interval $[0,1]$ as well as some basic notions and facts on fuzzy relations. The background is given in [6, 10, 18, 19].

For any $\{a_i\}_{i\in I} \subseteq [0,1]$ we shall denote $\bigvee_{i\in I} a_i = \sup\{a_i \mid i \in I\}$ and $\bigwedge_{i\in I} a_i = \inf\{a_i \mid i \in I\}$. In particular, for any $a, b \in [0,1]$, $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$. Then the interval $[0,1]$ becomes a complete distributive lattice.

A binary operation $*$ on $[a,b]$ is a t-norm if it is commutative, associative, non-decreasing in each argument and $a * 1 = a$ for all $a \in [0,1]$. With any continuous t-norm $*$ we associate its residuum:

$$a \to b = \bigvee\{a \in [0,1] \mid a * c \le b\}.$$

We list here the most well known continuous t-norms and their corresponding residua:
Łukasiewicz t-norm:

$$a *_L b = \max(0, a+b-1); \qquad a \to_L b = \min(1, 1-a+b)$$

Gödel t-norm:

$$a *_G b = a \wedge b; \qquad a \to_G b = \begin{cases} 1 & \text{if } a \le b \\ b & \text{if } a > b \end{cases}$$

Product t-norm:

$$a *_P b = ab; \qquad a \to_P b = \begin{cases} 1 & \text{if } a \le b \\ \frac{b}{a} & \text{if } a > b \end{cases}$$

A residuated lattice is a structure $(A, \vee, \wedge, *, \to, 0, 1)$ where $(A, \vee, \wedge, 0, 1)$ is a bounded lattice, $(A, *, \wedge)$ is an abelian monoid and $\to$ is a binary operation on A such that $a * b \le c$ iff $a \le b \to c$, for all $a, b, c \in A$.

The notion of BL-algebra was introduced by Hájek [18] as an abstraction of the structure $([0,1], \vee, \wedge, *, \to, 0, 1)$ where $*$ is a continuous t-norm and $\to$ is its residuum. A BL-algebra is a residuated lattice $(A, \vee, \wedge, *, \to, 0, 1)$ satisfying the following two conditions:

$$x * (x \to y) = x \wedge y \qquad \text{(divisibility axiom)}$$
$$(x \to y) \vee (y \to x) = 1 \qquad \text{(prelinearity axiom)}$$

A Gödel algebra is a BL-algebra satisfying the identity $x * x = x$. The Gödel t-norm $*_G = \wedge$ provides the Gödel algebra $([0,1], \vee, \wedge, *, \to, 0, 1)$.

The following two lemmas contain some useful properties of BL-algebras.

Lemma 2.1 ([6, 10, 18, 19]) *Let $(A, \vee, \wedge, *, \to, 0, 1)$ be a BL-algebra. For any $a, b, c \in A$ the following properties hold:*

(i) $a * (a \to b) = a \wedge b$;

(ii) $a \leq b \Leftrightarrow a \to b = 1$;

(iii) $a = 1 \to a$;

(iv) $1 = a \to a$;

(v) $a \to (b \to c) = (a * b) \to c = b \to (a \to c)$.

Lemma 2.2 ([6, 10, 18, 19]) *Let* $(A, \vee, \wedge, *, \to, 0, 1)$ *be a BL-algebra. For any* $\{a_i\}_{i \in I} \subseteq A$ *and* $a \in A$ *the following properties hold:*

(a) $(\bigvee_{i \in I} a_i) * a = \bigvee_{i \in I}(a_i * a)$;

(b) $a \to (\bigwedge_{i \in I} a_i) = \bigwedge_{i \in I}(a \to a_i)$;

(c) $(\bigvee_{i \in I} a_i) \to a = \bigwedge_{i \in I}(a \to a_i)$.

Another operation on a BL-algebra A is the biresiduum defined by

$$a \leftrightarrow b = (a \to b) \wedge (b \to a).$$

Let X be a non-empty set. A *fuzzy subset* of X is a function $A : X \to [0, 1]$. We denote by $\mathcal{P}(X)$ the family of crisp subsets of X and by $\mathcal{F}(X)$ the family of fuzzy subsets of X. If A is a crisp subset of X then its characteristic function $\chi_A : X \to \{0, 1\}$ is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

As a crisp subset of X is defined by its characteristic function, then we have $\mathcal{P}(X) \subseteq \mathcal{F}(X)$. A fuzzy subset A of X is non-zero if it is distinct from the characteristic function $\chi_\emptyset$ of the empty-set $\emptyset$. For any $A, B \in \mathcal{F}(X)$, by $A \subseteq B$ we mean that $A(x) \leq B(x)$ for each $x \in X$. A fuzzy subset A is *normal* if $A(x) = 1$ for some $x \in X$.

For $A, B \in \mathcal{F}(X)$ let us denote

$$I(A, B) = \bigwedge_{x \in X} (A(x) \to B(x))$$

and

$$E(A, B) = \bigwedge_{x \in X} (A(x) \leftrightarrow B(x)).$$

It is clear that $A \subseteq B$ iff $I(A, B) = 1$ and $A = B$ iff $E(A, B) = 1$. For any $x \in X$ we have

$$I(A, B) \leq A(x) \to B(x) \text{ and } E(A, B) \leq A(x) \leftrightarrow B(x).$$

The $I(A, B)$ is called *subsethood degree* of A and B and $E(A, B)$ the *degree of equality* (degree of similarity) of A and B. Intuitively $I(A, B)$ expresses the truth value of the statement "A is included in B" and $E(A, B)$ the truth value of the statement "A and B contain the same elements" (see [6, p. 82]).

A *fuzzy preference relation* Q on X is a function $Q : X^2 \to [0, 1]$, i. e. a fuzzy subset of X^2. The elements of X are interpreted as alternatives. If $x, y \in X$ then the real number $Q(x, y)$ represents the degree to which the alternative x is at least as good as y.

Let $*$ be a continuous t-norm. A fuzzy preference relation Q is said to be:

(a) *reflexive* if $Q(x, x) = 1$ for all $x \in X$;

(b) *symmetric* if $Q(x, y) = Q(y, x)$ for all $x, y \in X$;

(c) $*$-*transitive* if $Q(x, y) * Q(y, z) \leq Q(x, z)$ for all $x, y, z \in X$;

(d) *strongly total* if $Q(x, y) = 1$ or $Q(y, x) = 1$ for all distinct $x, y \in X$.

If it is not a danger of confusion, we will say transitive relation instead of $*$-transitive.

If a fuzzy preference relation Q is reflexive, symmetric and $*$-transitive then it is a *similarity relation* on X. Following [6, p. 190] for any fuzzy preference relation Q let us denote

$$\begin{aligned}
\mathrm{Ref}(Q) &= \bigwedge_{x \in X} Q(x, x); \\
\mathrm{Trans}(Q) &= \bigwedge_{x,y,z \in X} [(Q(x, y) * Q(y, z)) \to Q(x, z)]; \\
\mathrm{ST}(Q) &= \bigwedge_{x \neq y} (Q(x, y) \vee Q(y, x)).
\end{aligned}$$

$\mathrm{Ref}(Q)$ is called the degree of reflexivity of Q, $\mathrm{Trans}(Q)$ is called the degree of transitivity of Q and $\mathrm{ST}(Q)$ the degree of strong totality of Q.

Lemma 2.3 ([6]) *For any fuzzy preference relation Q the following equivalences hold:*

(i) Q is reflexive iff $\mathrm{Ref}(Q) = 1$*;*

(ii) Q is transitive if $\mathrm{Trans}(Q) = 1$*;*

(iii) Q is strongly total iff $\mathrm{ST}(Q) = 1$.

These indicators allow to compare two fuzzy preference relations with respect to a property; for example, if $\mathrm{Trans}(Q_1) \geq \mathrm{Trans}(Q_2)$, then the fuzzy preference relation Q_1 is "more transitive" than Q_2.

Lemma 2.4 ([6]) *Let Q be a fuzzy preference relation on X and $x, y, z \in X$. Then* $\mathrm{Trans}(Q) * Q(x, y) * Q(y, z) \leq Q(x, z)$.

3 Fuzzy Choice Functions

A vast literature has been dedicated to fuzzy choice functions and fuzzy preference relations (see [10, 20]). Most authors build their results on the thesis that social choice is governed by fuzzy preferences (hence modeled through fuzzy binary relations) but the act of choice is exact (hence choice functions are crisp) (see [3, 4, 5, 20]). They study crisp choice functions defined by fuzzy preference relations.

On the other hand, several situations in the world require to consider fuzzy choice functions [2, 10, 20, 22, 24, 34]. The first fuzzy choice function was emphasized by Orlovsky [22].

In [2] Banerjee admits the vagueness of the act of choice and develops a theory of revealed preference for choice functions with a fuzzy behaviour. He studies choice functions whose domain is the family of non-empty finite subsets of a universe of alternatives X and whose range consists of non-zero fuzzy subsets of X. Banerjee's thesis [2] is that "If preferences are permitted to be fuzzy, it seems natural to permit the choice functions to be fuzzy as well. This also tallies with experience". He studies in [2] a revealed preference for these fuzzy choice functions.

In [12, 13, 14, 15] we have considered fuzzy choice functions C for which the domain and the range are made of fuzzy subsets of X. Banerjee fuzzifies only the range of a choice function; we use a fuzzification of both the domain and the range of a choice function. In our case, the available sets of alternatives are fuzzy subsets of X. This leads to the introduction of the notion of availability degree of an alternative x with respect to an available set S. The availability degree might be useful when the decision maker possess partial information on the alternative x or when a criterion limits the possibility of choosing x. Therefore the available sets can be considered criteria in decision making.

We shall recall some notions and results in [12, 13, 14, 15, 16]. A fuzzy choice space is a pair $(X, \mathcal{B})$ where X is a non-empty set of alternatives and $\mathcal{B}$ is a non-empty family of non-zero fuzzy subsets of X. A fuzzy choice function on $(X, \mathcal{B})$ is a function $C : \mathcal{B} \to \mathcal{F}(X)$ such that for any $S \in \mathcal{B}$, $C(S)$ is a non-zero fuzzy subset of X and $C(S) \subseteq S$.

A lot of results in [12, 13, 14, 15, 16] require the following hypotheses:

($\mathbf{H}_1$) Every $S \in \mathcal{B}$ and $C(S)$ are normal fuzzy subsets of X;

($\mathbf{H}_2$) $\mathcal{B}$ includes the fuzzy sets $\chi_{\{x_1,\ldots,x_n\}}$ for any $n \geq 1$ and $x_1, \ldots, x_n \in X$.

For the crisp case ($\mathcal{B} \subseteq \mathcal{P}(X)$), $(\mathbf{H}_1)$ is automatically fulfilled in accordance with the definition of a (crisp) choice function, for the same case, $(\mathbf{H}_2)$ asserts that $\mathcal{B}$ includes all non-empty finite subsets of X. We fix a continuous t-norm $*$ and its residuum $\to$. Throughout this paper we shall assume that $(\mathbf{H}_1), (\mathbf{H}_2)$ are true.

If C is a fuzzy choice function on $(X, \mathcal{B})$ then we define two fuzzy preference relations R and $\bar{R}$ on X by:

$$R(x,y) = \bigvee_{S\in\mathcal{B}} (C(S)(x) \wedge S(y)),$$
$$\bar{R}(x,y) = C(\chi_{\{x,y\}})(x)$$

for all $x, y \in X$.

The fuzzy preference relations R and $\bar{R}$ defined above are fuzzy versions of crisp preference relations which have an important role in the classical revealed preference theory (see [26, 1, 24, 27], etc.). If $x, y \in X$, then the real number $R(x,y)$ is the degree of truth of the statement "there exists a criterion S such that alternative x is chosen with respect to S and alternative y verifies S". The $\bar{R}(x,y)$ represents the degree of truth of the statement "from the set $\{x,y\}$ is chosen at least the alternative x". We notice that the definition of the fuzzy preference relation $\bar{R}$ assumes that the fuzzy choice function C verifies hypothesis H_2.

Lemma 3.1 ([12]) *If C is a fuzzy choice function on $(X,\mathcal{B})$ then $\bar{R} \subseteq R$ and $R, \bar{R}$ are reflexive and strongly total.*

Let $(X,\mathcal{B})$ be a fuzzy choice space and Q be a fuzzy preference relation on X. For any $S \in \mathcal{B}$ let us define the fuzzy subset $G(S,Q)$ of X:

$$G(S,Q)(x) = S(x) * \bigwedge_{y\in X} [S(y) \rightarrow Q(x,y)]$$

for any $x \in X$. In this way one obtains a function $G(\cdot,Q) : \mathcal{B} \rightarrow \mathcal{F}(X)$.

Lemma 3.2 ([14]) *If C is a fuzzy choice function on $(X,\mathcal{B})$ then $C(S) \subseteq G(S,R)$ for any $S \in \mathcal{B}$.*

In general, $G(\cdot,Q)$ is not a fuzzy choice function: it might exist an $S \in \mathcal{B}$ such that $G(S,Q)$ is the zero set. According to Lemma 3.2, $G(\cdot,R)$ is a fuzzy choice function called the image of C.

A fuzzy choice function C on $(X,\mathcal{B})$ is said to be *rational* if $C = G(\cdot,Q)$ for some fuzzy preference relation Q on X. If Q is reflexive, transitive and strongly total, then C is called full rational.

Let C_1, C_2 be two fuzzy choice functions on $(X,\mathcal{B})$. Following [15, 16] the degree of similarity of C_1 and C_2 is defined by

$$E(C_1,C_2) = \bigwedge_{S\in\mathcal{B}} \bigwedge_{x\in X} [C_1(S)(x) \leftrightarrow C_2(S)(x)].$$

Lemma 3.3 ([16]) *For any fuzzy choice functions C_1, C_2 on $(X,\mathcal{B})$ we have:*

(i) $E(C_1,C_2) = 1$ *iff* $C_1 = C_2$;

(ii) $E(C_1,C_2) = E(C_2,C_1)$;

(iii) $E(C_1, C_2) * E(C_2, C_3) \leq E(C_1, C_3)$.

By this lemma, the function $(C_1, C_2) \mapsto E(C_1, C_2)$ is a similarity relation on the set of fuzzy choice functions defined on $(X, \mathcal{B})$. Similarity of fuzzy choice functions is a concept analogous to similarity of fuzzy preference relations ([6, 10, 18, 19]).

If we interpret X as a universe of alternatives and C_1, C_2 as the "choices" of two agents, then the real number $E(C_1, C_2)$ expresses how "similar" these choices are. In paper [15] the connection between similarity of two fuzzy choice functions and similarity of fuzzy preference relations associated with them is.

Lemma 3.4 ([16]) *Let C_1, C_2 be a fuzzy choice function on $(X, \mathcal{B})$. For any $S \in \mathcal{B}$ and $x \in X$ the following inequality holds:*

$$E(C_1, C_2) * C(S_1)(x) \leq C(S_2)(x).$$

Lemma 3.5 ([16]) *Let C be a fuzzy choice function on $(X, \mathcal{B})$ and Q be a fuzzy preference relation on X. Then $E(C, G(\cdot, Q)) * C(S)(s) * S(t) \leq Q(s, t)$ for all $S \in \mathcal{B}$ and $s, t \in X$.*

Proof. By Lemma 3.4 we have

$$E(C, G(\cdot, Q)) * C(S)(s) \leq G(S, Q)(x) \leq S(t) \to Q(s, t).$$

Hence according to the definition of the residuated lattice we get the desired inequality. ∎

Arrow's axiom (AA) is a condition introduced in [1] to characterize full rationality of crisp choice functions. We say that a crisp choice function C verifies (AA) if for any available sets of alternatives S_1, S_2 the following implication holds:

$$[S_1 \subseteq S_2] \Rightarrow [S_1 \cap C(S_2) = \emptyset] \text{ or } [S_1 \cap C(S_2) = C(S_1)].$$

Following [14], we say that a fuzzy choice function C on $(X, \mathcal{B})$ verifies the Fuzzy Arrow Axiom (FAA) if for all $S_1, S_2 \in \mathcal{B}$ and $x \in X$ we have

$$I(S_1, S_2) * S_1(x) * C(S_2)(x) \leq E(S_1 \cap C(S_2), C(S_1)).$$

It is easy to see that (FAA) is a fuzzy version of the Arrow Axiom (AA).

Intuitively, (FAA) says: "The maximum degree that, for some x, S_1 and S_2, x is chosen from S_2, x belongs to S_1 and S_1 is included in S_2, is less than or equal to the degree that the set of alternatives chosen from S_2, and also in S_1 is equal to the set of alternatives chosen from S_1" (cf. [16, p. 136]).

4 The Arrow Index

The Arrow axiom (AA) was introduced in [1] in order to characterize the full rationality of choice functions (for some information see also [31, pp. 20–30]).

These two properties have been later integrated by Sen [27] in a general result regarding the characterization of the rationality of a crisp choice function by various conditions (the axioms of revealed preference (WARP), (SARP), the congruence axioms (WCA), (SCA), etc.). This result is known as the Arrow–Sen theorem.

In [14] we extended the classical result of [1] by showing that (FAA) is equivalent with the full rationality of fuzzy choice functions. A new concept defined in [14] is the Arrow index $\mathcal{A}(C)$ of a fuzzy choice function C. The $\mathcal{A}(C)$ is a real number that expresses the degree to which C verifies the Fuzzy Arrow Axiom.

The main theorem of this section shows that $\mathcal{A}(C)$ characterizes the degree to which C is full rational. This type of result changes the perspective of studying the rationality of fuzzy choice functions. The attention is no longer focused on rational fuzzy choice functions, but the entire class of fuzzy choice functions has been taken into account with respect to their rationality. By the Arrow index, one can appreciate how rational any fuzzy choice function is. This fact has direct consequences in concrete problems. More often, the choices of an agent do not fulfill conditions of rationality or full rationality. By using the Arrow index, from a set of fuzzy choice functions we can select the ones with the maximum degree of rationality.

The results of this section offer a method for the calculation of the Arrow index in terms of the fuzzy preference relations R and $\bar{R}$. Let $(X, \mathcal{B})$ be a fuzzy choice space.

Definition 4.1 *Let C be a fuzzy choice function on $(X, \mathcal{B})$. The Arrow index $\mathcal{A}(C)$ of C is defined by*
$\mathcal{A}(C) = \bigwedge_{S_1,S_2 \in \mathcal{B}} \bigwedge_{x \in X} [(I(S_1, S_2) * S_1(x) * C(S_2)(x)) \rightarrow E(S_1 \cap C(S_2), C(S_1))]$.

We remark that $\mathcal{A}(C) = 1$ iff C verifies (FAA). Intuitively $\mathcal{A}(C)$ represents the degree of the statement "The fuzzy choice function C verifies the Fuzzy Arrow Axiom".

For the rest of this section we assume that $*$ is the minimum t-norm $\wedge$.

The following proposition expresses, in the language of indicators, the fact that the transitivity of the fuzzy preference relation $\bar{R}$ is ensured by the verification of the Arrow Axiom.

Proposition 4.2 *If C is a fuzzy choice function on $(X, \mathcal{B})$ then $\mathcal{A}(C) \leq Trans(\bar{R})$.*

Proof. Let $x, y \in X$. We shall prove that:

$$\mathcal{A}(C) \wedge C(\chi_{\{x,y\}})(x) \wedge C(\chi_{\{y,z\}})(y) \leq C(\chi_{\{x,y\}})(z). \tag{1}$$

Denote $S = \chi_{\{x,y,z\}}$. We shall establish the inequality

$$\mathcal{A}(C) \wedge C(\chi_{\{x,y\}})(x) \wedge C(\chi_{\{y,z\}})(y) \leq C(S)(x). \tag{2}$$

Assume by absurdum that

$$\mathcal{A}(C) \wedge C(\chi_{\{x,y\}})(x) \wedge C(\chi_{\{y,z\}})(y) \not\leq C(S)(x). \tag{3}$$

We shall prove that (3) implies the condition

$$\mathcal{A}(C) \wedge C(\chi_{\{x,y\}})(x) \wedge C(\chi_{\{y,z\}})(y) \not\leq C(S)(y). \tag{4}$$

Assume that (4) does not hold, i. e.

$$\mathcal{A}(C) \wedge C(\chi_{\{x,y\}})(x) \wedge C(\chi_{\{y,z\}})(y) \leq C(S)(y). \tag{5}$$

We remark that $I(\chi_{\{x,y\}}, S) = 1$ and $\chi_{\{x,y\}}(y) = 1$, hence

$$\begin{gathered}
\mathcal{A}(C) \wedge C(S)(y) = \mathcal{A}(C) \wedge I(\chi_{\{x,y\}}, S) \wedge \chi_{\{x,y\}}(y) \wedge C(S)(y) \leq \\
\leq I(\chi_{\{x,y\}}, S) \wedge \chi_{\{x,y\}}(y) \wedge C(S)(y) \wedge \\
[I(\chi_{\{x,y\}}, S) \wedge \chi_{\{x,y\}}(y) \wedge C(S)(y) \to E(\chi_{\{x,y\}} \cap C(S), C(\chi_{\{x,y\}}))].
\end{gathered}$$

Therefore by using Lemma 2.1 (i) we get

$$\begin{aligned}
\mathcal{A}(C) \wedge C(S)(y) &\leq E(\chi_{\{x,y\}} \cap C(S), C(\chi_{\{x,y\}})) \\
&\leq I(C(\chi_{\{x,y\}}, \chi_{\{x,y\}} \cap C(S)) \\
&\leq C(\chi_{\{x,y\}}(x) \to (\chi_{\{x,y\}} \wedge C(S)(y)) \\
&\leq C(\chi_{\{x,y\}}(x) \to C(S)(x))
\end{aligned}$$

By definition of residuated lattice it follows that

$$\mathcal{A}(C) \wedge C(S)(y) \wedge C(\chi_{\{x,y\}})(x) \leq C(S)(x).$$

In accordance with (5)

$$\begin{aligned}
&\mathcal{A}(C) \wedge C(\chi_{\{x,y\}})(x) \wedge C(\chi_{\{y,z\}})(y) \\
&\qquad \leq \mathcal{A}(C) \wedge C(S)(y) \wedge C(\chi_{\{x,y\}})(x) \leq C(S)(x)
\end{aligned}$$

contradicting (3). Then we obtain (4). Similarly we get

$$\mathcal{A}(C) \wedge C(\chi_{\{x,y\}})(x) \wedge C(\chi_{\{x,y\}})(y) \not\leq C(S)(z). \tag{6}$$

From (3), (4) and (6) we infer that $C(S)(x) \neq 1$, $C(S)(y) \neq 1$ and $C(S)(z) \neq 1$, contradicting that $C(S)$ is a normal fuzzy subset of X (cf. $\mathbf{H}_1$). This contradiction shows that (2) holds. We remark that

$$\begin{gathered}
\mathcal{A}(C) \wedge C(S)(x) = \mathcal{A}(C) \wedge I(\chi_{\{x,z\}}, S) \wedge \chi_{\{x,y\}}(x) \wedge C(S)(x) \leq \\
I(\chi_{\{x,z\}}, S) \wedge \chi_{\{x,z\}}(x) \wedge C(S)(x) \wedge \\
[(I(\chi_{\{x,z\}}, S) \wedge \chi_{\{x,z\}}(x) \wedge C(S)(x)) \to E(\chi_{\{x,z\}} \cap C(S), C(\chi_{\{x,z\}}))] \leq \\
E(\chi_{\{x,z\}} \cap C(S)) \leq ([x,y] \cap C(S))(x) \to C(\chi_{\{x,z\}})(x) \leq \\
C(S)(x) \to C(\chi_{\{x,z\}})(x)
\end{gathered}$$

hence we obtain

$$\mathcal{A}(C) \wedge C(S)(x) \leq C(\chi_{\{x,z\}})(x).$$

Using (2) it follows that

$$\mathcal{A}(C) \wedge C(\chi_{\{x,y\}})(x) \wedge C(\chi_{\{y,z\}})(y) \leq \mathcal{A}(C) \wedge C(S) \leq C(\chi_{\{x,z\}})(x).$$

Then for all $x, y, z \in X$ we have

$$\mathcal{A}(C) \leq (C(\chi_{\{x,y\}})(x) \wedge C(\chi_{\{y,z\}})(y)) \to C(\chi_{\{x,z\}})(x) = \\ (\bar{R}(x,y) \wedge \bar{R}(y,z)) \to \bar{R}(x,z)$$

therefore

$$\mathcal{A}(C) \leq \bigwedge_{x,y,z \in X} [(\bar{R}(x,y) \wedge \bar{R}(y,z)) \to \bar{R}(x,z)) \leq \mathrm{Trans}(\bar{R}).$$

■

Corollary 4.3 *If C satisfies (FAA) then $\bar{R}$ is transitive.*

Proof. By Proposition 4.2, $\mathcal{A}(C) = 1$ implies $\mathrm{Trans}(\bar{R}) = 1$. ■

Let us denote by $\mathcal{R}$ the family of all fuzzy preference relations on X.

Theorem 4.4 *If C is a fuzzy choice function on $(X, \mathcal{B})$ then*

$$\mathcal{A}(C) = \bigvee_{Q \in \mathcal{R}} [E(C, G(\cdot, Q)) \wedge \mathrm{Trans}(Q) \wedge \mathrm{Ref}(Q) \wedge \mathrm{ST}(Q)] = \\ = E(C, G(\cdot, \bar{R})) \wedge \mathrm{Trans}(\bar{R}).$$

Proof. Let $Q \in \mathcal{R}$. We shall prove that

$$E(C, G(\cdot, Q)) \wedge \mathrm{Trans}(Q) \wedge \mathrm{Ref}(Q) \wedge \mathrm{ST}(Q) \leq \mathcal{A}(C). \quad (7)$$

Let $S_1, S_2 \in \mathcal{B}$ and $x \in X$. We shall establish the inequality

$$E(C, G(\cdot, Q)) \wedge \mathrm{Trans}(Q) \wedge \mathrm{Ref}(Q) \wedge \mathrm{ST}(Q) \leq \\ \leq (I(S_1, S_2)) \wedge S_1(x) \wedge C(S_2))(x) \to E(S_1 \cap C(S_2)), C(S_1)). \quad (8)$$

Let us denote

$$\alpha = E(C, G(\cdot, Q)) \wedge \mathrm{Trans}(Q) \wedge \mathrm{Ref}(Q) \wedge \mathrm{ST}(Q) \wedge I(S_1, S_2)) \wedge S_1(x) \wedge C(S_2)(x).$$

By the definition of residuated lattice, the inequality (8) is equivalent to the inequality

$$\alpha \leq E(S_1 \cap C(S_2), C(S_1)). \quad (9)$$

Let $z, v \in X$. Then by Lemmas 3.5 and 2.4 we have

$$\begin{aligned}
&\alpha \wedge I(S_1, S_2) \wedge S_1(x) \wedge C(S_2)(x) \wedge C(S_1)(z) \wedge S_2(v) \leq \\
&\alpha \wedge [E(C, G(\cdot, Q)) \wedge C(S_1)(z) \wedge S_1(x)] \wedge [E(C, G(\cdot, Q)) \wedge C(S_1)(x) \wedge S_2(v)] \leq \\
&\qquad \alpha \wedge Q(z, x) \wedge Q(z, v) \leq \mathrm{Trans}(Q) \wedge Q(z, x) \wedge Q(x, v) \leq Q(z, v).
\end{aligned}$$

By Lemma 2.1 we get

$$\alpha \wedge C(S_1)(z) = \alpha \wedge I(S_1, S_2) \wedge S_1(x) \wedge C(S_2)(x) \wedge C(S_1)(z) \leq S_2(v) \to Q(z, v)$$

for all $v \in X$, hence

$$\alpha \wedge C(S_1)(z) \wedge \bigwedge_{v \in X} [S_2(v) \to Q(z, v)].$$

On the other hand,

$$\begin{aligned}
&\alpha \wedge C(S_1)(z) \leq I(S_1, S_2) \wedge S_1(x) \wedge C(S_2)(x) \wedge C(S_1)(z) \leq \\
&\qquad C(S_1)(z) \wedge I(S_1, S_2) \leq S_1(z) \wedge [S_1(z) \to S_2(z)] \leq S_2(z).
\end{aligned}$$

Therefore

$$\alpha \wedge C(S_1)(z) \leq S_2(z) \wedge \bigwedge_{v \in X} [S_2(v) \to Q(z, v)] = G(S_2, Q)(z).$$

Since $\alpha \wedge C(S_1)(z) \leq \alpha \leq E(C, G(\cdot, Q))$ by Lemma 3.4 we obtain

$$\alpha \wedge C(S_1)(z) \wedge E(C, G(\cdot, Q)) \wedge G(S_2, Q)(z) \leq C(S_2)(z),$$

hence by Lemma 2.1(iii) and (ii) it follows that

$$\begin{aligned}
&\alpha \leq C(S_1)(z) \to C(S_2)(z) = [C(S_1)(z) \to S_1(z)] \wedge [C(S_1)(z) \to C(S_2)(z)] = \\
&\qquad C(S_1)(z) \to (S_1(z) \wedge C(S_2)(z)). \quad (10)
\end{aligned}$$

Let $y, z \in X$. According to Lemma 2.1(i), Lemma 3.5 and Lemma 2.4 the following inequality holds:

$$\begin{aligned}
&\alpha \wedge S_1(z) \wedge C(S_2)(z) \wedge S_1(y) = \\
&\quad = \alpha \wedge I(S_1, S_2) \wedge S_1(x) \wedge C(S_2)(x) \wedge S_1(z) \wedge C(S_2)(z) \wedge S_1(y) \leq \\
&\quad \leq S_1(y) \wedge S_2(y) \wedge C(S_2)(x) \wedge C(S_2)(z) = \\
&\quad = \alpha \wedge S_1(y) \wedge S_2(y) \wedge C(S_2)(x) \wedge C(S_2)(z) \leq \\
&\alpha \wedge [E(C, G(\cdot, Q)) \wedge C(S_2)(z) \wedge S_2(x)] \wedge [E(C, G(\cdot, Q))] \wedge C(S_2)(x) \wedge S_2(y)] \leq \\
&\quad \leq \alpha \wedge Q(z, x) \wedge Q(x, y) \leq \mathrm{Trans}(Q) \wedge Q(z, x) \wedge Q(x, y) \leq Q(z, y).
\end{aligned}$$

Thus we obtain: $\alpha \wedge S_1(z) \wedge C(S_2)(z) \leq S_1(y) \to Q(z, y)$ for all $y \in X$, hence

$$\alpha \wedge S_1(z) \wedge C(S_2)(z) \leq S_1(z) \wedge \bigwedge_{y \in X} [S_1(y) \to Q(z, y)] = G(S_1, Q)(z).$$

Since $\alpha \leq E(C, G(\cdot, Q))$, by Lemma 3.4 we get

$$\alpha \wedge S_1(z) \wedge C(S_2)(z) \leq E(C, G(\cdot Q)) \wedge G(S_1, Q)(z) \leq C(S_1)(z).$$

Therefore we get for all $z \in X$:

$$\alpha \leq (S_1(z) \wedge C(S_2)(z)) \rightarrow C(S_1)(z). \tag{11}$$

Combining (10) and (11) it follows that

$$\alpha \leq \bigwedge_{z \in X} [(S_1(z) \wedge C(S_2)(z)) \leftrightarrow C(S_1)(z)] = E(S_1 \cap C(S_2), C(S_1)).$$

Thus (9) was proved, hence (8) is verified. Since (8) holds for all $S_1, S_2 \in \mathcal{B}$ and $x \in X$ we obtain (7). From (7) we can derive the inequality:

$$\bigvee_{Q \in \mathcal{R}} [E(C, G(\cdot, Q)) \wedge \mathrm{Trans}(Q) \wedge \mathrm{Ref}(Q) \wedge \mathrm{ST}(Q)] \leq \mathcal{A}(C). \tag{12}$$

Now we shall prove the inequality

$$\mathcal{A}(C) \leq E(C, G(\cdot, \bar{R})). \tag{13}$$

Let $S \in \mathcal{B}$ and $x \in X$. We shall establish the following two inequalities:

$$\mathcal{A}(C) \leq C(S)(x) \rightarrow G(S, \bar{R}) \tag{14}$$
$$\mathcal{A}(C) \leq G(S, \bar{R})(x) \rightarrow C(S)(x). \tag{15}$$

Let $z \in X$. Thus we have

$$\begin{aligned} \mathcal{A}(C) \wedge C(S)(x) \wedge S(z) = \mathcal{A}(C) \wedge (I(\chi_{\{x,z\}}, S) \wedge \chi_{\{x,z\}}(x) \wedge C(S)(x)) \leq \\ \leq I(\chi_{\{x,z\}}, S) \wedge \chi_{\{x,z\}}(x) \wedge C(S)(x) \wedge \\ [(I[x,z], S) \wedge \chi_{\{x,z\}}(x) \wedge C(S)(x)) \rightarrow E(\chi_{\{x,z\}} \cap C(S), C(\chi_{\{x,z\}})] = \\ = I(\chi_{\{x,z\}}, S) \wedge \chi_{\{x,z\}}(x) \wedge C(S)(x) \wedge E(\chi_{\{x,z\}} \cap C(S), C(\chi_{\{x,z\}})] \leq \\ \leq E(\chi_{\{x,z\}} \cap C(S), C(\chi_{\{x,z\}}) \leq (\chi_{\{x,z\}}(x) \wedge C(S)(x) \leftrightarrow C(\chi_{\{x,z\}})(x)) = \\ = C(S)(x) \leftrightarrow \bar{R}(x,z) \leq C(S)(x) \rightarrow \bar{R}(x,z), \end{aligned}$$

hence

$$\mathcal{A}(C) \wedge C(S)(x) \wedge S(z) \leq \bar{R}(x,z).$$

Therefore $\mathcal{A}(C) \wedge C(S)(x) \leq S(z) \rightarrow \bar{R}(x,z)$ for each $z \in X$, hence

$$\mathcal{A}(C) \wedge C(S)(x) \leq S(x) \wedge \bigwedge_{z \in X} [S(z) \rightarrow \bar{R}(x,z)] = G(S, \bar{R})(x).$$

Thus $\mathcal{A}(C) \wedge C(S)(x) \rightarrow G(S, \bar{R})(x)$, so (14) holds. In order to prove (15) we consider an element $y \in X$ such that $C(S)(y) = 1$ (because $C(S)$ is a normal fuzzy subset of X), then $S(y) = 1$. Thus

$$G(S, \bar{R})(x) \leq S(y) \rightarrow \bar{R}(x,y) = 1 \rightarrow \bar{R}(x,y) = \bar{R}(x,y) = C(\chi_{\{x,y\}})(x).$$

We remark that

$$G(S,\bar{R})(x) \leq S(x) = S(x) \wedge S(y) \wedge C(S)(y) = I(\chi_{\{x,y\}}, S) \wedge \chi_{\{x,y\}}(y) \wedge C(S)(y),$$

therefore

$$\begin{aligned}
&\mathcal{A}(C) \wedge G(S,\bar{R}) \leq \\
&(I(\chi_{\{x,y\}}, S) \wedge \chi_{\{x,y\}}(y) \wedge C(S)(y)) \wedge [(I(\chi_{\{x,y\}}, S) \wedge \chi_{\{x,y\}}(y) \wedge C(S)(y)) \rightarrow \\
&\qquad E(\chi_{\{x,y\}} \cap C(S), C(\chi_{\{x,y\}}))] \leq E(\chi_{\{x,y\}} \cap C(S), C(\chi_{\{x,y\}})).
\end{aligned}$$

Since $G(S,\bar{R})(x) \leq C(\chi_{\{x,y\}})(x)$ we obtain

$$\begin{aligned}
\mathcal{A}(C) \wedge G(S,\bar{R})(x) &\leq C(\chi_{\{x,y\}}, x) \wedge E(\chi_{\{x,y\}} \cap C(S), C(\chi_{\{x,y\}})) \leq \\
&\leq C(\chi_{\{x,y\}})(x) \wedge [(\chi_{\{x,y\}})(x) \wedge C(S)(x) \rightarrow C(\chi_{\{x,y\}})(x)] = \\
&= \bar{R}(x,y) \wedge [C(S)(x) \rightarrow \bar{R}(x,y)] \leq \\
&\leq \bar{R}(x,y) \wedge [\bar{R}(x,y) \rightarrow C(S)(x)] = \bar{R}(x,y) \wedge C(S)(x) \leq C(S)(x)
\end{aligned}$$

hence $\mathcal{A}(C) \leq G(S,\bar{R})(x) \rightarrow C(S)(x)$ and (15) is proved.

The inequalities (14) and (15) hold for all $S \in \mathcal{B}$ and $x \in X$ hence

$$\mathcal{A}(C) \leq \bigwedge_{S\in\mathcal{B}} \bigwedge_{x\in X} [C(S)(x) \leftrightarrow G(S,\bar{R})(x)] = E(C, G(\cdot,\bar{R})).$$

The inequality (13) was proved. Since $\bar{R}$ is reflexive and strongly total, we have $\mathrm{Ref}(\bar{R}) = \mathrm{ST}(\bar{R}) = 1$. Thus, by Proposition 4.2 and (13) we get

$$\begin{aligned}
&\mathcal{A}(C) \leq E(C, G(\cdot,\bar{R})) \wedge \mathrm{Trans}(\bar{R}) = \\
&\qquad E(C, G(\cdot,\bar{R})) \wedge \mathrm{Trans}(\bar{R}) \wedge \mathrm{Ref}(\bar{R}) \wedge \mathrm{ST}(\bar{R}) \leq \\
&\qquad\qquad \bigvee_{Q\in\mathcal{R}} [E(C, G(\cdot,Q)) \wedge \mathrm{Trans}(Q) \wedge \mathrm{Ref}(Q) \wedge \mathrm{ST}(Q)].
\end{aligned}$$

■

Remark 4.5 *By Theorem 4.4 the following assertions are equivalent:*

(a) C satisfies (FAA);

(b) $E(C, G(\cdot,Q)) = \mathrm{Trans}(Q) = \mathrm{Ref}(Q) = \mathrm{ST}(Q) = 1$ for some $Q \in \mathcal{R}$;

(c) C is full rational.

Therefore we have obtained the main result of [14]: (FAA) is equivalent with the full rationality.

Let us denote by W the transitive closure of the fuzzy revealed preference relation R (see [6] or [10]). In [12] there have been introduced the following congruence axioms for fuzzy choice functions:

(WFCA) (*Weak Fuzzy Congruence Axiom*)

For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds:

$$R(x,y) \wedge C(S)(y) \wedge S(x) \leq C(S)(x).$$

(SFCA) (*Strong Fuzzy Congruence Axiom*)

For any $S \in \mathcal{B}$ and $x, y \in X$ the following inequality holds:

$$W(x,y) \wedge C(S)(y) \wedge S(x) \leq C(S)(x).$$

(WFCA), (SFCA) are fuzzy versions of the classical congruence axioms (WCA), (SCA) [24, 27, 30, 31].

In [16] the following indicators of the axioms (WFCA) and (SFCA) have been defined.

Definition 4.6 ([16]) *For a fuzzy choice function C on a fuzzy choice space $(X, \mathcal{B})$ let us define the indicators:*

(i) $\mathrm{WFCA}(C) = \bigwedge_{x,y \in X} \bigwedge_{S \in \mathcal{B}} [(S(x) \wedge C(S)(y) \wedge R(x,y)) \rightarrow C(S)(x)]$

(ii) $\mathrm{SFCA}(C) = \bigwedge_{x,y \in X} \bigwedge_{S \in \mathcal{B}} [(S(x) \wedge C(S)(y) \wedge W(x,y)) \rightarrow C(S)(x)]$.

Remark 4.7 *The following assertions are equivalent:*

(i) $\mathrm{WFCA}(C) = 1$ *iff C verifies the axiom (WFCA);*

(ii) $\mathrm{SFCA}(C) = 1$ *iff C verifies the axiom (SFCA).*

The indicator $\mathrm{WFCA}(C)$ (resp. $\mathrm{SFCA}(C)$) expresses the degree to which the fuzzy choice function C satisfies the axiom (WFCA) (resp. (SFCA)).

We recall the following theorem from [16, p. 214].

Theorem 4.8 *If C is a fuzzy choice function then*

$$\begin{aligned} \mathrm{WFCA}(C) = \mathrm{SFCA}(C) = E(C, G(\cdot, \bar{R})) \wedge \mathrm{Trans}(R) \\ = E(C, G(\cdot, \bar{R})) \wedge \mathrm{Trans}(\bar{R}). \end{aligned}$$

By combining Theorems 4.4 and 4.8 it follows that:

Theorem 4.9 *If C is a fuzzy choice function then*

$$\begin{aligned} \mathcal{A}(C) = \mathrm{WFCA}(C) = \mathrm{SFCA}(C) = E(C, G(\cdot, R)) \wedge \mathrm{Trans}(R), \\ = E(C, G(\cdot, \bar{R})) \wedge \mathrm{Trans}(\bar{R})). \end{aligned}$$

Remark 4.10 *The fuzzy Arrow-Sen theorem established in [11, 16] is a particular case of Theorem 4.9. In case when C is a crisp choice function, one obtains an important part of the classic Arrow-Sen theorem ([1, 27, 31]).*

Remark 4.11 *The proofs of the results from this section used essentially the fact that $*$ is the minimum t-norm $\wedge$ (for example, the idempotence of $\wedge$).*

5 Concluding Remarks

The treatment of rationality in fuzzy revealed preference theory emphasizes the connection between the fuzzy choice functions and the fuzzy preference relations which rationalize them. This interconnection is realized by conditions on fuzzy choice functions (axioms of revealed preference and congruence, consistency properties, etc.) Such condition related to the fuzzy choice functions is the Fuzzy Arrow Axiom (FAA), a generalization in fuzzy context of the well known axiom of Arrow [1].

The main theorem of [14] shows the equivalence between (FAA) and the full rationality of the fuzzy choice function. The notion of full rationality combines two types of properties:

(a) properties of fuzzy choice functions: rationality;

(b) properties of fuzzy preference relations: reflexivity, transitivity, strong completeness.

In this paper the relation between (FAA) and the full rationality has been analyzed from another perspective:

(a) instead of the Arrow Axiom, one took the Arrow index $\mathcal{A}(C)$ of a fuzzy choice function, indicator which expresses the degree to which C verifies (FAA);

(b) one considered a numerical measure of the full rationality, expressed by the degree of similarity of fuzzy choice functions and by the indicators of reflexivity, transitivity and strong completeness of the fuzzy preference relation.

The equivalence between (FAA) and the full rationality is here transformed into the equality between $\mathcal{A}(C)$ and a real number which expresses the degree of full rationality.

The main theorem of the paper (Theorem 4.4) consists of the equality of three numerical indicators, and the result of [14] is one if its particular cases. By combining Theorem 4.4 with a result of [16, p. 214] on the congruence indicators WFCA(C) and SFCA(C), one obtains their equality with the Arrow index, and an evaluation of their value in terms of the similarity and the transitivity indicator.

The result of this paper strengthens the one in [14] for the following two reasons:

(i) it consists of a numerical relation valid for any fuzzy choice function;

(ii) it provides a method of ranking the fuzzy choice functions with respect to their rationality.

It remains to investigate to which extent the theory of this paper can be developed without hypotheses ($\mathbf{H}_1$) and ($\mathbf{H}_2$). At the same time, an open problem is to investigate the validity of these results in the more general context offered by a left-continuous t-norm or for other particular t-norms.

The notions from this paper can be obviously defined in the context of the L-valued fuzzy sets, where L is a residuated lattice [6], and some results remain true assuming that L is a Gödel algebra.

References

[1] K.J. Arrow, Rational choice functions and orderings, Economica, 26, 1959, 121–127

[2] A. Banerjee, Fuzzy choice functions, revealed preference and rationality, Fuzzy Sets Syst., 70, 1995, 31–43

[3] C. R. Barrett, P. K. Pattanaik, M. Salles, On choosing rationally when preferences are fuzzy, Fuzzy Sets Syst., 34, 1990, 197–212

[4] C. R. Barrett, P. K. Pattanaik, M. Salles, Rationality and aggregation of preferences in an ordinally fuzzy framework, Fuzzy Sets Syst., 49, 1992, 9–13

[5] C. R. Barrett, P. K. Pattanaik, M. Salles, On the structure of fuzzy social welfare functions, Fuzzy Sets Syst., 19, 1996, 1–10

[6] R. Bělohlávek, Fuzzy relational systems. Foundations and principles, Kluwer, 2002

[7] M. Dasgupta, R. Deb, Fuzzy choice functions. Soc Choice Welfare, 8, 1991, 171–182

[8] B. De Baets, J. Fodor, Twenty years of fuzzy preference relations (1978–1997), Belgian Journal of Operations Research, Statistics and Computer Science, 37, 1997, 61–82

[9] B. De Baets, J. Fodor (Eds.), Principles of fuzzy preference modelling and decision making, Academia Press, Gent, 2003

[10] J. Fodor, M. Roubens, Fuzzy preference modelling and multicriteria decision support, Kluwer, Dordrecht, 1994

[11] I. Georgescu, Rational choice and revealed preference: a fuzzy approach, Turku Centre for Computer Science PhD Dissertation 60, 2005

[12] I. Georgescu, On the axioms of revealed preference in fuzzy consumer theory, Journal of Systems Science and Systems Engineering, 13, 2004, 279–296

[13] I. Georgescu, Consistency conditions in fuzzy consumers theory, Fundamenta Informaticae, 61, 2004, 223–245

[14] I. Georgescu, Arrow's axiom and full rationality for fuzzy choice functions, Social Choice and Welfare, 28, 2007, 303–319

[15] I. Georgescu, Similarity of fuzzy choice functions, Fuzzy Sets Syst., 158, 2007, 1314–1326

[16] I. Georgescu, Fuzzy choice functions: a revealed preference approach, Berlin, Springer, 2007

[17] S. Klaue, K. Kurbel, I. Loutschko, Automated negotiations on agent–based e–marketplaces: An overview. Proceedings of the 14th Bled Electronic Commerce Conference, Bled, Slovenia, 508–519, 2001

[18] P. Hájek, Metamathematics of fuzzy logic, Kluwer, 1998

[19] E. P. Klement, R. Mesiar, E. Pap, Triangular norms, Kluwer, 2000

[20] P. Kulshreshtha, B. Shekar, Interrelationship among fuzzy preference - based choice function and significance of rationality conditions: a taxonomic and intuitive perspective, Fuzzy Sets Syst., 109, 2000, 429–445

[21] K. Kurbel, I. Loutschko, Towards multi–agent electronic marketplaces: What is there and what is missing? The Knowledge Engineering Review, 18, 2003, 33–46

[22] S. Orlovsky, Decision–making with a fuzzy preference relation, Fuzzy Sets Syst., 1, 1978, 155–167

[23] S. Ovchinnikov, Decision–making with a ternary binary relation, in Proceedings of the Conference on Information Processing and Management of Uncertainty IPMU 2004, Perugia, Italy, 511–516

[24] M. Richter, Revealed preference theory, Econometrica, 34, 1966, 635–645

[25] M. Roubens, Some properties of choice functions based on valued binary relations, Eur. J. Op. Res., 40, 1989, 309–321

[26] P.A. Samuelson, A note of the pure theory of consumer's behavior, Economica, 5, 1938, 61–71

[27] A.K. Sen, Choice functions and revealed preference, Review of Economic Studies, 38, 1971, 307–312

[28] A.K. Sen, Social choice theory: A re–examination. Econometrica, 45, 1977, 53–89

[29] A.K. Sen, Choice, welfare and measurement. Cambridge MA: MIT Press and Oxford: Blackwell, 1982

[30] K. Suzumura, Rational choice and revealed preference, Review of Economic Studies, 43, 1976, 149–159

[31] K. Suzumura, Rational choice, collective decisions and social welfare, Cambridge University Press. Cambridge, 1983

[32] E. Turunen, Mathematics behind fuzzy logic, Physica-Verlag, 1999

[33] B. Van de Walle, S. Heitch, P. Faratin, Coping with one–to–many multicriteria negotiations in electronic marketplaces: beyond price discovery, Munich, Germany, 2001

[34] Ph. De Wilde, Fuzzy utility and equilibria, IEEE Transactions on Systems, Man and Cybernetics, 34, 2004, 1774–1785

[35] H. Uzawa, A note on preference and axioms of choice, Annals of the Institute of Statistical Mathematics, 8, 1956, 35–40

[36] L. A. Zadeh, Similarity relations and fuzzy orderings, Inf. Sci. 3, 1971, 177–200