Does the New Keynesian Model Have a Uniqueness Problem?*

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Abstract

This paper addresses whether non-uniqueness of equilibrium is a substantive problem for policy analysis in New-Keynesian (NK) models. There would be a substantive problem if there were no compelling way to select among different equilibria that give different answers to critical policy questions. In fact there is: stability-under-learning. We focus our analysis on the efficacy of fiscal policy when the economy is in the ZLB. We study a fully non-linear NK model with Calvo-pricing frictions and argue that the model has a unique stable-under-learning rational expectations equilibrium. In that equilibrium,

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the implications of the model for fiscal policy inherit all of the key properties of linearized NK models. We also find that for empirically plausible cases, linear approximations work quite well for assessing the size of the government spending multiplier and the drop in GDP that occurs in the ZLB.

1. Introduction

New Keynesian (NK) models have been enormously influential in terms of their policy implications[1]. The models' implications for fiscal policy are particularly striking when the zero lower bound (ZLB) on the nominal rate of interest is binding.[2] Eggertsson and Woodford (2003) (EW) and Eggertsson (2004) develop an elegant and transparent framework for studying fiscal policy in the NK model at the ZLB.

The key results that emerge from the literature can be summarized as follows[3]. First, when the ZLB binds, the fall in output is potentially very large. Second, the output multiplier associated with government consumption is larger when the ZLB binds than when it does not bind. Third, the more flexible are prices and the longer is the expected duration of the ZLB is longer, the larger is the drop in output and the larger is the government consumption multiplier.

These controversial results are based on literature that uses a linearized version of the NK model, which has a unique solution. In fact, the non-linear NK models have multiple equilibria, even if one restricts attention, as did EW, to minimum state variable ZLB equilibria. As stressed by Mertens and Ravn (2015), policy prescriptions can vary a great deal across those equilibria. At some ZLB equilibria, the

[1] For a classic exposition of the NK model see Woodford (2003.)

[2] It is widely understood that zero is not the critical lower bound. What is critical is that some lower bound on the interest becomes binding on monetary policy.

[3] see, for example, EW, Eggertsson (2011) and Christiano, Eichenbaum and Rebelo (2011) (CER),

government consumption multiplier is small or even negative. In others, it is very large. So, in principle, non-uniqueness of equilibria poses an enormous challenge for policy analysis based on NK models.

This paper addresses a simple question: is non-uniqueness of equilibria a *substantive* problem for policy analysis in NK models? There would be a substantive problem if there were no compelling way to select among different equilibria that give different answers to critical policy questions. To be concrete we focus our analysis on the impact of changes in government consumption when the economy is in the ZLB.

Our argument starts from the presumption that the assumption of rational expectations is *obviously* wrong. But it can be a useful modeling strategy for thinking about a world where the strong assumptions associated with rational expectations aren't literally satisfied. Indeed that is how Lucas viewed it.

"... the model describe above 'assumes' that agents know a great deal about the structure of the economy and perform some non-routine computations. It is in order to ask, then: will an economy with agents armed with 'sensible' rules-of-thumb, revising these rules from time to time so as to claim observed rents, tend as time passes to behave as described..." Lucas (1978)

In the spirit of the literature summarized by Evans and Honkapohja (2001), we adopt the following selection criterion for rational expectations equilibria (REE). Suppose agents make a 'small' error in forming expectations about variables relative to their values in a particular REE. Would the economy converge to a REE, if agents form expectations using simple learning rules? If yes, then we say the REE is stable-under-learning, or for short, learnable. From this perspective, stability-under-learning is a necessary condition for an REE and the associated policy implications to be empirically interesting. REE equilibria that aren't learnable are best treated as mathematical curiosities.

We apply this stable-under-learning criterion to a standard fully non-linear NK model with Calvo pricing frictions. Working with this model poses two interesting challenges. First, unlike linearized NK models of the type considered by EW, the ZLB REE can't be characterized by a set of numbers. Because there is an endogenous state variable (past price dispersion), the ZLB REE is a set of functions. Second, we must think about how agents might learn about these functions.

Our basic results can be summarized as follows. First, consistent with Mertens and Ravn (2015) we find that there are multiple REE, including sunspot equilibria. When we consider fundamental shocks that trigger ZLB episodes, we find two minimum state variable ZLB equilibria. These equilibria converge to different inflation rates if the ZLB episode lasts forever. Second, like Mertens and Ravn (2015), we find that impact of government consumption can be very different in the different ZLB equilibria. For example, there exist both sunspot and minimum state ZLB REE in which the government consumption multiplier is actually negative. Third, we argue that there exists a unique interior ZLB equilibrium in the non-linear Calvo model that is stable-under-learning. Fourth, and most importantly, the controversial predictions of the linearized NK model about fiscal policy in the ZLB, including the large size of the government consumption multiplier at the ZLB are satisfied at the unique learnable ZLB REE. That equilibrium is the one that converges to a relatively low ZLB deflation rate. Based on this analysis we conclude that the Calvo model does not have a substantive uniqueness problem, as least for the analysis of fiscal policy in the ZLB.

Many authors have used non-linear versions of the Rotemberg (1982) model of nominal price rigidities to proxy for the Calvo model. In the Rotemberg model the representative firm faces a quadratic cost of adjusting nominal prices. It is well known that linear approximations to the Calvo and the Rotemberg models give rise to the same set of equations whose solution defines an REE. In contrast, non-linear versions of the model are potentially very different. As it turns out some of the non-linear properties of the Rotemberg model are very sensitive to the details of how one formulates adjustment costs for prices. Specifically, we show that the number of rational expectations ZLB equilibria and their stability properties depends on whether and exactly how one scales adjustment costs for growth. Remarkably, we still always find that there exists a unique ZLB REE that is stable-under-learning. Moreover, all of the predictions of the log-linear NK for the impact of fiscal policy in the ZLB hold at that equilibrium. Indeed, for our benchmark parametrizations, the value of the government consumption multiplier in the linear and non-linear model are remarkably similar.

As a by-product of our analysis, we use our non-linear model to assess the robustness of policy implications about fiscal policy at the ZLB that have been derived using log linear approximations to the NK model. We find that linear approximations work quite well for assessing the size of the government spending multiplier and the drop in GDP that occurs in the ZLB. Evidence that the quality of linear approximations is poor rests on examples where output deviates by more than roughly 20 percent from its steady state, cases where no one would expect linear approximations to work well. There is one interesting difference between the linear and non-linear models. It is well know that for some parameters values, the multiplier in the linear model shoots off to infinity, say as the expected length of the ZLB episode becomes large or prices become very flexible (see for example CER (2011)). For the same parameter values, these extreme results manifest themselves in a different way in the non-linear Calvo model: a ZLB REE simply ceases to exist.

The Great Recession was a very unusual event. So the learning equilibrium underlying our stability calculations are of interest as a way of modeling how agents behaved in the wake of a shock that pushes the economy into a prolonged ZLB episode. So we analyze the impact of an increase in government consumption along the learning equilibrium that converges to the stable ZLB REE. Our findings here can be summarized as follows. First, the learning equilibrium is unique. Second, the size of the multiplier is large in the learning equilibrium. The latter finding is different than results reported in Mertens and Ravn (2015). As it turns out the main reason for the difference in our results is that despite their backwards looking learning rule, Mertens and Ravn change agents expectations about future consumption and inflation when they change government consumption. We do not.

The remainder of this paper is organized as follows. In section 2 we discuss multiplicity and learnability in the context of a standard flexible price model. We do so in order to define learnability in a very simple environment and contrast it with the notion of stability of a REE employed by Benhabib, Schmidt-Gorhe and Uribe (2001). In section three we analyze ZLB REE in a nonlinear Calvo model. We also assess the quality of linear approximations to the Calvo model in this section. Section four contains our main results regarding stability-under-learning of different ZLB REE. In section five we discuss learning equilibrium. Section six contains our analysis of the non-linear Rotemberg model. Concluding remarks are contained in section seven.

2. Equilibrium Selection in a Flexible Price Model

In this section we discuss multiplicity and learnability in the context of a standard flexible price model. Our analysis is closely related to that of Evans, Guse, and Honkapojha (EGH) (2008) who analyze similar issues in an New Keynesian (NK) model. Our primary objective is to define learnability in a very simple environment and contrast it with the notion of stability employed by Benhabib, Schmidt-Gorhe and Uribe (2001). Consistent with EGH (2008), we show analytically that there is a unique learnable equilibrium in the model. As is well know, this equilibrium casts doubt with neo-Fischerian interpretations of the low, post Great Recession, rate of inflation.

2.1. Model Economy

As in BSGU (2001) we consider an endowment economy populated by a large number of identical infinitely lived households with preferences defined over consumption and real balances. For simplicity we assume that these preferences are separable between consumption and real balances. The representative household maximizes

$$\sum_{j=0}^{\infty} \beta^j \left[u(C_{t+j}) + v\left(\frac{M_{t+j}}{P_{t+j}}\right)\right].$$

The household receives a constant endowment of the consumption good and faces the budget constraint

$$C_t + \tau_t + \frac{B_t}{P_t} + \frac{M_t}{P_t} \leq (1 + R_{t-1}) \frac{B_{t-1}}{P_t} + Y + \frac{M_{t-1}}{P_t}.$$

Here P_t, C_t, τ_t and Y denote the price level, consumption, lump-sump taxes and the endowment at time t. The variables B_t and M_t denote the end of time t holdings of one-period nominal bonds and money, respectively which $R_{t-1}$is the nominal interest on a bond held at the end of time $t-1$.

The household chooses paths for consumption, real money balances, and bond holding so that the following optimality conditions hold

$$u'(C_t) = v'\left(\frac{M_t}{P_t}\right) + \beta\frac{u'(C_{t+1})}{\pi_{t+1}}$$

$$u'(C_t) = \beta\frac{R_t}{\pi_{t+1}}u'(C_{t+1})$$

In equilibrium, $C_t = Y$, so that we obtain:

$$1 = \frac{v'\left(\frac{M_t}{P_t}\right)}{u'(Y)} + \frac{\beta}{\pi_{t+1}}$$

$$1 = \beta\frac{R_t}{\pi_{t+1}}$$

Monetary policy given by a Taylor, subject to a ZLB constraint on the nominal interest rate:

$$R_t = R(\pi_t) = \max\left\{1, \frac{\pi^*}{\beta} + \alpha(\pi_t - \pi^*)\right\}. \tag{2.1}$$

Here π^* is monetary authority's target rate of inflation. We assume $\alpha > 1$ so that the so-called Taylor principle is satisfied. The presence of the max operator reflects the ZLB.

There is a block-recursive structure to the equilibrium in which one can solve for π_t and R_t and M_t/P_t is determined according to agents' demand for real balances. Specifically, we first use the consumption Euler equation to solve for a sequence of inflation rates, given π_0, that satisfy

$$\pi_{t+1} = \beta R(\pi_t) \tag{2.2}$$

We then solve for R_t using (2.1). Finally, given the sequence of equilibrium inflation rates we solve for the sequence of real money balances that makes the following equilibrium condition hold[4]

$$v'\left(\frac{M_t}{P_t}\right) = u'(Y)\left[\beta\frac{R(\pi_t)}{\pi_{t+1}} - \frac{\beta}{\pi_{t+1}}\right]. \tag{2.3}$$

It is well known that in this type of model the equilibrium value of π_0 is indeterminate. But given an assumed value for π_0, the remainder of the equilibrium can be constructed as discussed.

BSGU note if $\alpha > \frac{1}{\beta}$, then there are two steady states to the system. The first steady state, which we refer to as the high-inflation steady state, has the property that $\pi_t = \pi^*$. The second steady state, which we refer to as the low-inflation steady state, has the property that $\pi_t = \beta$.

2.2. Stability in BSGU (2001)

BSGU study the stability properties of these steady states and argue that only low-inflation state is stable. By stability they mean the following. Suppose that π_t close to but not exactly equal to either π^* or β. A steady state is stable if the economy converges back to it. In a rational expectations equilibrium, agents' beliefs about future inflation coincide with actual inflation rates. So BSGU iterate equation (2.2) forward, and analyze to which, if any steady state, the economy converges to.

There are three cases to consider. In the first case, $\pi_0 < \frac{\beta - \pi^*}{\alpha\beta} + \pi^*$. Equation (2.2) implies that

$$\pi_1 = \max\left\{\beta, \pi^* + \alpha\beta\left(\pi_0 - \pi^*\right)\right\} = \beta.$$

and $\pi_t = \beta$ for $t > 1$. In the second case, $\frac{\beta - \pi^*}{\alpha\beta} + \pi^* < \pi_0 < \pi^*$. Then equation (2.2)

[4] We assume that $v(\cdot)$ has whatever properties are required for there to exist a sequence of real money balances that satisfy (2.3).

implies that

$$\pi_1 = \max\left\{\beta, \pi^* + \alpha\beta\left(\pi_0 - \pi^*\right)\right\} < \pi_0.$$

It follows that $\pi_t \leq \max\left\{\beta, \pi^* + (\alpha\beta)^t(\pi_0 - \pi^*)\right\}$ which in turn implies that $\pi_t \rightarrow \beta$. In the third case, $\pi_0 > \pi^*$, so that

$$\pi_1 = \pi^* + \alpha\beta\left(\pi_0 - \pi^*\right).$$

It follows that $\pi_t \geq \pi^* + (\alpha\beta)^t\left(\pi_0 - \pi^*\right)$ which in turn implies that $\pi_t \rightarrow \infty$. In this simple model, there is no reason to rule out explosive inflation paths.[5]

Two key observations follow from the previous analysis. First, the high-inflation steady state is not stable, in the sense that BSGU use that term. That is, for small deviations of π_0 from π^*, the equilibrium path for π_t either converges to β or ∞. Second, the low-inflation steady state is stable as long as the initial rate of inflation does not exceed π^*.

The previous stability results have been used to justify a neo-Fisherian view of monetary policy. Specifically, authors like Bullard (2013, 2015) use the stability of the negative-inflation steady state to argue that at a commitment to a low nominal interest rate leads to a low inflation rate.[6] Here the term 'lower' refers to the relative values of a variable across steady states. The stability argument can also be viewed as an argument for why many advanced economies have experienced low inflation for so long (see Bullard (2015)). The idea is that, since the high-inflation rate steady state equilibrium isn't stable, a small deviation from it could lead countries like Japan into the stable low inflation steady state.

[5] There exist various modifications to the simple interest-rate rule considered here that can rule out some of these equilibria. See for example Christiano and Rostagno (2001) and Atkeson, Chari, and Kehoe (2010).

[6] Bullard (2013) makes the argument in a simple NK model.

2.3. Stability-under-learning

We now contrast the stability in the BSGU sense, with stability-under-learning. Suppose that in a neighborhood of a steady state, agents have expectations about future inflation that aren't equal to the steady value of inflation. Denote by $\pi_{t+1|t}$ agents' time t belief about the value of $\pi_{t+1.}$ We assume agents update their beliefs according to a well defined learning rule. A learning equilibrium is the sequence of prices and quantities that obtains under that learning rule. A rational expectations equilibrium is said to be *stable-under learning* if the learning equilibrium converges back to the RE equilibrium.

In what follows we assume that agents update their beliefs according to the adaptive learning rule[7]

$$\pi_{t+1|t} = \pi_{t-1}.$$

We solve for the learning equilibrium in the following way. For a given belief, $\pi_{t+1|t}$, inflation in the current period is determined so that the household Euler equation holds and the monetary policy rule is satisfied:

$$\pi_{t+1|t} = \beta R(\pi_t) = \max\{\beta, \pi^* + \alpha\beta(\pi_t - \pi^*)\}.$$

For an initial belief, $\pi_{1|0}$, that is close to, but not equal to, π^*

$$\pi_0 = \pi^* + \frac{\pi_{1|0} - \pi^*}{\alpha\beta}$$

meaning that

$$|\pi_1 - \pi^*| < |\pi_{1|0} - \pi^*|.$$

[7] Our results are robust to allowing for constant gain learning or least squares learning.

Given the assumed learning process, $\pi_{2|1} = \pi_0$. For every $t > 0$, we then have that $|\pi_t - \pi^*| < |\pi_{t-1} - \pi^*|$ and that that $\pi_t \to \pi^*$. So the high-inflation rational expectations equilibrium is stable under learning.

Consider an initial belief, $\pi_{1|0}$, that is close to, but slightly larger than β. Then

$$\pi_0 = \pi^* + \frac{\pi_{1|0} - \pi^*}{\alpha\beta}$$

meaning that $\pi_1 > \beta$ and that

$$|\pi_1 - \pi^*| < |\pi_{1|0} - \pi^*|.$$

Given the assumed learning process, $\pi_{2|1} = \pi_0$. For every $t > 0$, we then have that $|\pi_t - \pi^*| < |\pi_{t-1} - \pi^*|$ and that that $\pi_t \to \pi^*$. So the low-inflation rational expectations equilibrium is not stable under learning. Given the lower bound on nominal interest rates, the household Euler equations implies that a belief about inflation in the next period that has $\pi_{t+1|t} < \beta$ cannot be rationalized so we do not consider this case.

The previous discussion illustrates the sharp contrast between stability under learning and stability in the BSGU sense. We reach exactly the opposite conclusions about the "stability" of the two steady state equilibria. The low-inflation steady state equilibrium is unique stable steady state in the BSGU sense. The high-inflation steady state equilibrium is the unique stable steady state under learning. Given this result, our view is that only the high inflation steady state is empirically interesting. Finally, note that the learning equilibrium leading to the high inflation steady state is unique.

From the learning perspective, the low-inflation steady state is empirically uninteresting. It seems incredible to think about the US or Japan about being in the low inflation steady state when any perturbation of beliefs would have sent the economy away from that point. We use stability-under-learning as the equilibrium selection device.

3. Fiscal Policy in the ZLB

In this section we assess the implications of the NK model for fiscal policy at the ZLB. The goal of our analysis is to assess whether the model has robust implications for fiscal multipliers once we impose stability under learning as an equilibrium selection device. We conduct our analysis in a non-linear version of the NK model. In this paper, we restrict ourselves to minimum-state-variable (MSV) equilibria (the type of equilibria considered in EW). We conduct our analysis using a fully non-linear version of the NK model in which firms face Calvo price setting frictions. Authors like Christiano and Eichenbaum (2012) and Braun, Boneva, and Waki (2015) interpret the price frictions in their nonlinear analysis of the NK model as stemming from adjustment costs as proposed by Rotemberg (1982). This interpretation is interesting because it implies the same linearized equations that EW study. The advantage of adopting Rotemberg adjustment costs is analytic simplicity. The Calvo approach injects an endogenous state variable (past price dispersion), while there is no endogenous state variable in the Rotemberg approach. However, as we show in Section 6, there are some important pitfalls associated with using the Rotemberg model that arise from its sensitivity to how the costs of adjusting prices is formulated.

3.1. Model Economy

A representative household maximizes

$$E_0 \sum_{t=0}^{\infty} d_t \left[\log \left(C_t \right) - \frac{\chi}{2} h_t^2 \right]$$

where C_t denotes consumption, h_t denotes hours work, and

$$d_t = \prod_{j=0}^{t} \left(\frac{1}{1 + r_{j-1}} \right).$$

As in EW, we assume that r_t can take on two values: r and r^ℓ, where $r^\ell < 0$. The stochastic process for r_t is given by

$$\Pr \left[r_{t+1} = r^\ell | r_t = r^\ell \right] = p, \;\; \Pr \left[r_{t+1} = r | r_t = r^\ell \right] = 1 - p, \;\; \Pr \left[r_{t+1} = r^\ell | r_t = r \right] = 0.$$

We assume that r_t is known at time t. The household faces the budget constraint

$$P_t C_t + B_t \leq \left(1 + R_{t-1} \right) B_{t-1} + W_t h_t + \Pi_t.$$

Here P_t is the price of the consumption good, B_t denotes the household's nominal risk-free bond holdings, R_{t-1} is the gross nominal interest rate paid on bonds held from period $t-1$ to period t, W_t is the nominal wage, and Π_t represents lump-sum profits net of lump-sum government taxes.

The two first order necessary conditions associated with an interior solution to the household's problem are:

$$\chi h_t C_t = \frac{W_t}{P_t} \tag{3.1}$$

$$\frac{1}{1+R_t} = \frac{1}{1+r_t} E_t \frac{P_t C_t}{P_{t+1} C_{t+1}}. \quad (3.2)$$

A final homogeneous good, Y_t, is produced by competitive and identical firms using the following technology:

$$Y_t = \left[\int_0^1 (Y_{j,t})^{\frac{\varepsilon}{\varepsilon-1}} \, dj \right]^{\frac{\varepsilon-1}{\varepsilon}}, \quad (3.3)$$

where $\varepsilon > 1$. The representative firm chooses specialized inputs, $Y_{j,t}$, to maximize profits:

$$P_t Y_t - \int_0^1 P_{j,t} Y_{j,t} dj,$$

subject to the production function (3.3). The firm's first order condition for the j^{th} input is:

$$Y_{j,t} = (P_t / P_{j,t})^{-\varepsilon} Y_t. \quad (3.4)$$

The j^{th} input good in (3.3) is produced by a firm j who is a monopolist in the product market and is competitive in factor markets. Monopolist j has the production function:

$$Y_{j,t} = h_{j,t}. \quad (3.5)$$

Here $h_{j,t}$ is the quantity of labor used by the j^{th} monopolist. The monopolist maximizes

$$E_t \sum_{k=0}^{\infty} \beta^k \lambda_{t+k} \left((1+\upsilon) \tilde{P}_t - P_{t+k} s_{t+k} \right) Y_{j,t+k} \quad (3.6)$$

by choosing $\tilde{P}_t$. Here υ is a subsidy designed to remove steady state distortions owing to monopoly power. The j^{th} retailer sets its price, $P_{j,t}$, subject to the demand curve, (3.4), and the following Calvo sticky price friction (3.7):

$$P_{j,t} = \begin{cases} P_{j,t-1} & \text{with probability } \theta \\ \tilde{P}_t & \text{with probability } 1-\theta \end{cases} . \tag{3.7}$$

The firm satisfies whatever demand occurs at its posted price. The real marginal cost facing each monopolist is given by:

$$s_t \equiv \frac{W_t}{P_t} = \chi h_t C_t. \tag{3.8}$$

The first order condition of monopolist j can be written as

$$\tilde{p}_t = \pi_t \frac{K_t}{F_t}$$

where $\tilde{p}_t \equiv \frac{\tilde{P}_t}{P_{t-1}}$,

$$K_t = \frac{Y_t}{C_t} s_t + \theta \frac{1}{1+r_t} E_t \pi_{t+1}^{\varepsilon} K_{t+1}$$

and

$$F_t = \frac{Y_t}{C_t} + \theta \frac{1}{1+r_t} E_t \pi_{t+1}^{\varepsilon-1} F_{t+1}.$$

Here π_t denotes the gross rate of inflation.

It is well known that aggregate output can be written as[8]

$$Y_t = p_t^* h_t$$

[8] See for example Woodford (2003).

where p_t^* is a measure of price dispersion, which evolves according to

$$p_t^* = \left[(1-\theta) \left[\frac{1-\theta \pi_t^{\varepsilon-1}}{1-\theta} \right]^{\frac{-\varepsilon}{1-\varepsilon}} + \theta \pi_t^{\varepsilon} (p_{t-1}^*)^{-1} \right]^{-1}.$$

The aggregate resource constraint is given by

$$C_t + G_t \leq Y_t. \tag{3.9}$$

In equilibrium, this constraint is satisfied as an equality because households and government go to the boundary of their budget constraints. Government consumption is an exogenous process discussed below.

Monetary policy rule is given by

$$R_t = \max \{1, 1 + r + \alpha (\pi_t - 1)\} \tag{3.10}$$

As above, the max operator reflects the zero lower bound constraint on nominal interest rates and α is assumed to be larger than $1 + r$.

We assume that $r_0 = r^{\ell}$. To complete our specification of the environment, we note that there are no other shocks to the model. We consider two scenarios. In the first, the government does not respond to the discount–rate shock. In the second, G_t increases by one percent of steady state output as long as $r_t = r^{\ell}$.

3.2. Solving the Non-Linear Calvo Model

The price dispersion term, p_{t-1}^* is the only state variable in our system other than the exogenous discount factor shock. It is convenient to collect the equilibrium conditions of the model:

$$
\begin{aligned}
p_t^* &= \left[(1-\theta) \left[\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta} \right]^{\frac{\varepsilon}{\varepsilon-1}} + \theta\pi_t^{\varepsilon}(p_{t-1}^*)^{-1} \right]^{-1} \qquad (3.11) \\
\frac{1}{Y_t - G_t} &= \frac{1}{1+r_t} \max\left(1, 1+r+\alpha\left(\pi_t - 1\right)\right) E_t \frac{1}{Y_{t+1} - G_{t+1}} \frac{1}{\pi_{t+1}} \\
F_t &= \frac{Y_t}{Y_t - G_t} + \theta \frac{1}{1+r_t} E_t \pi_{t+1}^{\varepsilon-1} F_{t+1} \\
F_t \left[\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}} &= \chi \frac{Y_t^2}{p_t^*} + \theta \frac{1}{1+r_t} E_t \pi_{t+1}^{\varepsilon} F_{t+1} \left[\frac{1-\theta\pi_{t+1}^{\varepsilon-1}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}}
\end{aligned}
$$

A solution to the model is a set of functions $Y(p_{t-1}^*, r_t)$, $\pi(p_{t-1}^*, r_t)$, $F(p_{t-1}^*, r_t)$, $p^*(p_{t-1}^*, r_t)$ which satisfy the four equilibrium conditions (3.11).

In the first stage we solve for the equilibrium functions that obtain when $r_t = r$, i.e. after the economy has exited the ZLB. As in Bizer and Judd (1989) we begin with a conjectured set of equilibrium functions, $\tilde{Y}(\cdot), \tilde{\pi}(\cdot), \tilde{F}(\cdot), \tilde{p}^*(\cdot)$, for the time $t+1$ variables that appear in (3.11). The equilibrium conditions give us a mapping

$$
\left[Y(\cdot), \quad \pi(\cdot), \quad F(\cdot), \quad p^*(\cdot) \right] = T\left[\tilde{Y}(\cdot), \tilde{\pi}(\cdot), \tilde{F}(\cdot), \tilde{p}^*(\cdot) \right].
$$

At a rational expectations equilibrium

$$
\left[Y(\cdot), \quad \pi(\cdot), \quad F(\cdot), \quad p^*(\cdot) \right] = T\left[Y(\cdot), \pi(\cdot), F(\cdot), p^*(\cdot) \right].
$$

We approximate these functions using finite elements methods on a grid defined over p_{t-1}^*. Given a value of p_{t-1}^*, and the conjectured set of equilibrium functions, (3.11) reduces to a systems of four equations in four unknowns, Y_t, π_t, F_t and p_t^*. We solve these equations for all of the values of p_{t-1}^* in the grid. In this way we constructs a function from the state variable, p_{t-1}^* to the equilibrium quantities. If the functions that are produced are the same as the conjectured equilibrium functions, then we have found an equilibrium. If they aren't the same, then we use the newly computed functions as conjectured equilibrium functions and repeat the process until the approximating functions converge.

In a second stage, we solve for the equilibrium functions that obtain when r_t is equal to r_ℓ. Define

$$\begin{aligned} Y_\ell(p^*_{t-1}) &= Y(p^*_{t-1}, r_\ell),\ p^*_\ell(p^*_{t-1}) = p^*(p^*_{t-1}, r_\ell), \\ F_\ell(p^*_{t-1}) &= F(p^*_{t-1}, r_\ell),\ \pi_\ell(p^*_{t-1}) = \pi(p^*_{t-1}, r_\ell). \end{aligned}$$

In the ZLB, we can write (3.11) as

$$p^*_\ell(p^*_{t-1}) = \left[(1-\theta)\left[\frac{1-\theta\pi_\ell(p^*_{t-1})^{\varepsilon-1}}{1-\theta}\right]^{\frac{\varepsilon}{\varepsilon-1}} + \theta\frac{\pi_\ell(p^*_{t-1})^{\varepsilon}}{p^*_{t-1}}\right]^{-1}$$

$$\begin{aligned} \frac{1}{Y_\ell(p^*_{t-1}) - G_\ell} = \frac{1}{1+r^l}\max\left(1 + r + \alpha\left(\pi_\ell(p^*_{t-1}) - 1\right), 1\right)&\left[p\frac{1}{Y_\ell(p^*_t) - G_\ell}\frac{1}{\pi_\ell(p^*_t)}\right. \\ &\left. + (1-p)\frac{1}{Y(p^*_t) - G}\frac{1}{\pi(p^*_t)}\right] \end{aligned}$$

$$F_\ell(p^*_{t-1}) = \frac{Y_\ell(p^*_{t-1})}{Y_\ell(p^*_{t-1}) - G_\ell} + \theta\frac{1}{1+r^l}\left[p\pi_\ell(p^*_t)^{\varepsilon-1}F_\ell(p^*_t) + p\pi(p^*_t)^{\varepsilon-1}F(p^*_t)\right]$$

$$
\begin{aligned}
F_\ell(p_{t-1}^*)\left[\frac{1-\theta\pi_\ell(p_{t-1}^*)^{\varepsilon-1}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}} =& \chi Y_\ell(p_{t-1}^*)(Y_\ell(p_{t-1}^*)/p_\ell^*(p_{t-1}^*)) \\
&+\theta\frac{1}{1+r^l}p\pi_\ell(p_t^*)^{-\varepsilon}F_\ell(p_t^*)\left[\frac{1-\theta\pi_\ell(p_t^*)^{\varepsilon-1}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}} \\
&+\theta\frac{1}{1+r^l}(1-p)\pi(p_t^*)^{-\varepsilon}F(p_t^*)\left[\frac{1-\theta\pi(p_t^*)^{\varepsilon-1}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}
\end{aligned}
$$

We solve for the equilibrium functions $Y_\ell(p_{t-1}^*), \pi_\ell(p_{t-1}^*)$, $F_\ell(p_{t-1}^*)$ and $p_\ell^*(p_{t-1}^*)$ using the same algorithm used in the first stage.

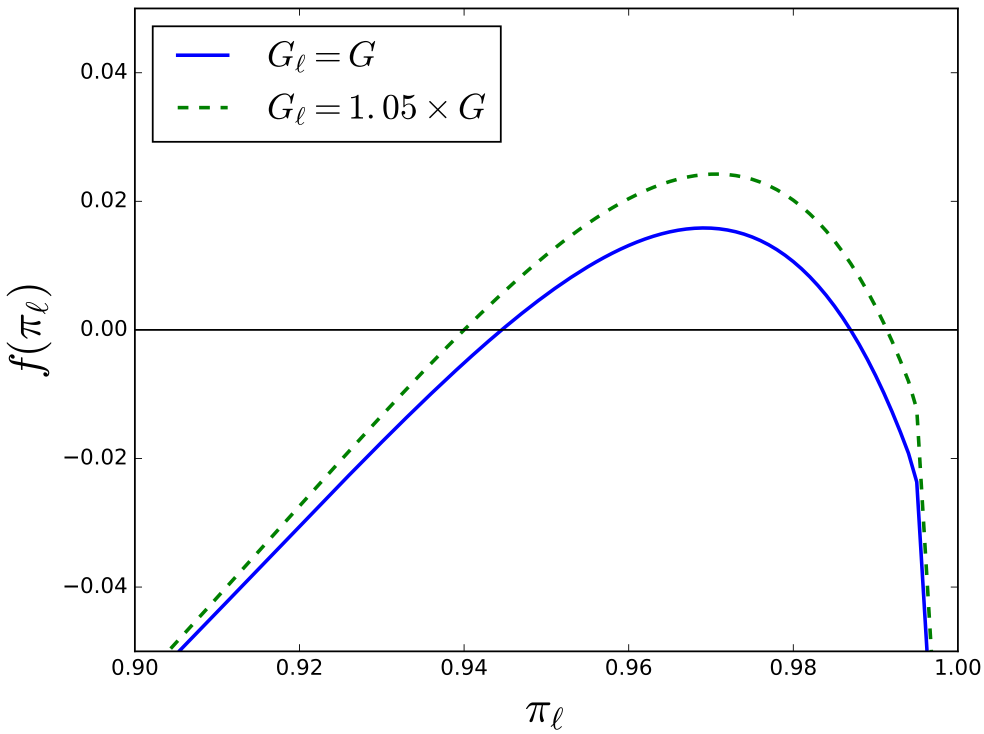
One important feature of the solution, as pointed out by Mertens and Ravn (2015), is the limit as $t \to \infty$ when the economy stays in the ZLB. In this case, p_t^* converges to a number, $\hat{p}$, for any interior equilibrium. At such a limiting point, the above system of equations collapses to a system of equations in four unknowns, $\pi_\ell(\hat{p})$, $Y_\ell(\hat{p})$, $p_\ell^*(\hat{p})$, and $F_\ell(\hat{p})$. We refer to the set of the limiting equilibrium values of prices and quantities as the steady-state ZLB equilibrium.

We solve for a steady-state ZLB equilibrium as follows. Conjecture a guess for $\pi_\ell(\hat{p})$. Then calculate the implied value of $\hat{p}$ from the first equation, calculate $C_\ell(\hat{p})$ from the second equation and compute $F_\ell(\hat{p})$ from the third equation. Then check if the final equation holds with equality. If it holds, $\pi_\ell(\hat{p})$ is a steady-state ZLB equilibrium value of inflation. If it doesn't hold, search for another $\pi_\ell(\hat{p})$. Employing this algorithm, we make the equations defining an interior steady-state ZLB equilibrium collapse into one equation one unknown

$$f(\pi_\ell) = 0. \tag{3.12}$$

In a slight abuse of notation we have dropped the explicit dependence of π_ℓ on $\hat{p}$. If this condition did not hold, then π_ℓ couldn't be the steady-state ZLB equilibrium

Figure 3.1: Steady State ZLB Equilibrium Function



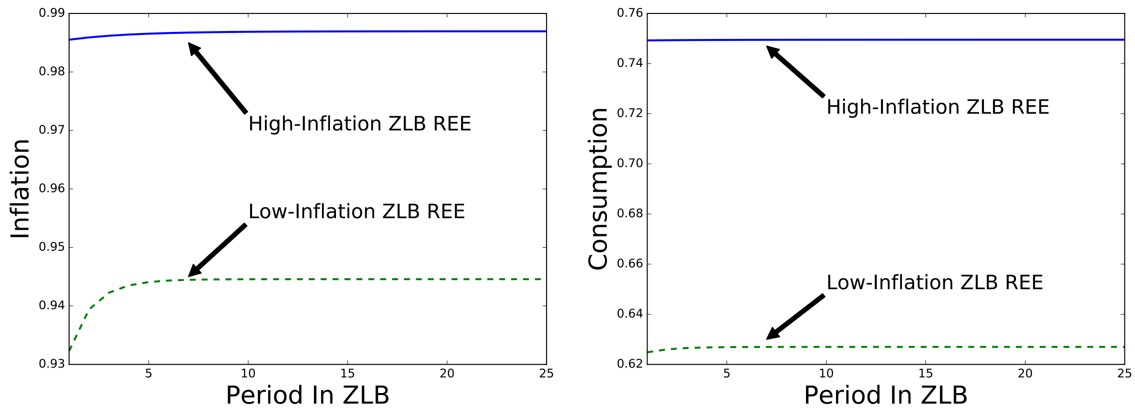
value of inflation. So, a necessary condition for the ZLB equilibrium to be unique is that there is a unique solution to (3.12).

In our experiments we use following baseline parameterization of the model:

$$\begin{aligned} \varepsilon &= 7.0,\ \beta = 0.99,\ \alpha = 2.0,\ p = 0.75, \\ r^\ell &= -0.02/4,\ \theta = 0.85,\ \eta_g = 0.2. \end{aligned} \tag{3.13}$$

Steady state output is normalized to 1 by setting $\chi = 1.25$.

Figure 3.2: RE Equilibrium Paths In ZLB

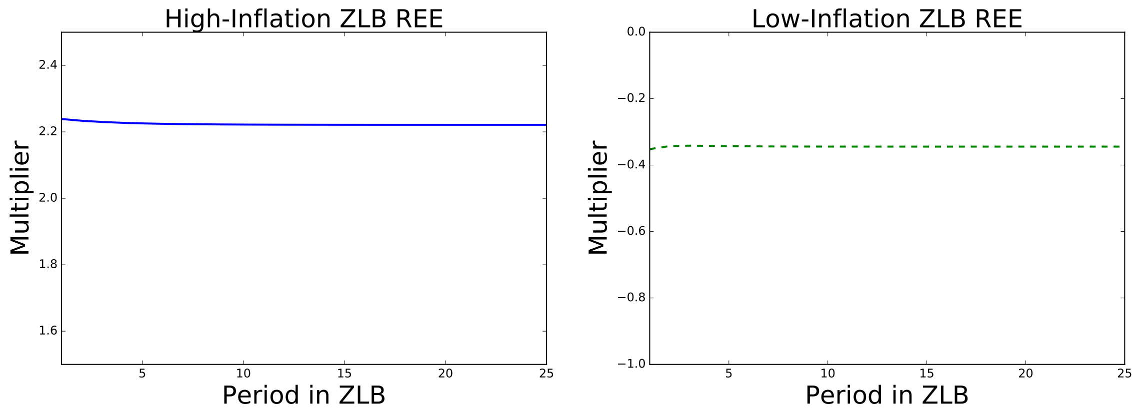


3.3. Baseline RE Equilibria Results

Recall that the equation defining an interior steady-state ZLB equilibrium is given by (3.12). Figure 3.1, displays $f(\pi_\ell)$ as a function of π_ℓ. The solid line is calculated assuming that G is equal to its steady state value, 0.20. Note that $f(\pi_\ell)$ has an inverted U shape. It follows that there are either two interior steady-state ZLB equilibria or no such equilibria. Given our assumed parameter values, there are two steady-state ZLB equilibria which we refer to as the high and low inflation steady state ZLB equilibria. In practice we find that the number of ZLB equilibria coincides with the number of steady state ZLB equilibria. To be clear, this is a numerical result, not a theorem

The dotted lines of the panels of Figure 3.2 displays the dynamic response of inflation and consumption, respectively, to the discount rate shock when the economy converges to the high and low inflation steady-state ZLB equilibrium. We refer to these paths as the high and low inflation ZLB equilibria. A number of features are worth noting. First, along the high inflation ZLB equilibrium path, quarterly

Figure 3.3: RE Multiplier In ZLB



inflation and consumption drop in the impact period of the shock by 1.5 and -6.35 percentage points, respectively. After about 5 quarters these declines stabilize at -1.3 and -6.3 percentage points, respectively. Second, along the low inflation ZLB equilibrium path, quarterly inflation and consumption drop in the impact period of the shock by -7.25 and -23.5 percentage points, respectively. After about 5 quarters these declines stabilize at -6.0 and -23.3 percentage points, respectively.

To derive values for the multiplier we assume that $G^{\ell} = 1.05 \times G^{h}$, i.e. when the economy is in the ZLB, G rises by 1 percentage of steady state output. We define the multiplier in the first period to be

$$\frac{G^{\ell}}{(C^{\ell}(p^{*}_{t-1}) + G^{\ell})} \frac{\Delta \left(C^{\ell}(p^{*}_{t-1}) + G^{\ell} \right)}{\Delta G^{\ell}}.$$

We compute this ratio assuming that if the economy is in the high inflation (low inflation) ZLB equilibrium for a low value of G, it is in the high inflation (low inflation) ZLB equilibrium for the high value of G.[9] The two panels of Figure 3.3 display

[9]This assumption is non-trivial because one can easily instruct examples in which G serves as a

the government spending multiplier in the high inflation and low inflation ZLB equilibrium as a function of time. Notice that the multiplier in the high inflation ZLB equilibrium is large, exceeding two over the time period displayed. In contrast, the multiplier is actually *negative* in the low inflation ZLB equilibrium. To understand this result, note that an increase in G^{ℓ} shifts f upwards (see Figure 3.1). This shift implies that the effect of an increase in G depends on which equilibrium we focus on. This change in sign is a dramatic illustration of the basic result in Mertens and Raven (2011) where the multiplier is much lower in the analog to our low inflation ZLB equilibrium.

3.3.1. Comparisons to linearized version of the model

Table 3.1 summarizes our results regarding the impact of changes in G for the non-linear and linear versions of the Calvo model. We report the response of inflation, output and the multiplier in the impact period of a shock to the discount rate accompanied by a rise in G. Notice that the equilibrium response of the linearized model are similar to the high-inflation ZLB equilibrium. For example, the impact multiplier in the linear model is 1.63 while it is 2.24 in the high-inflation ZLB equilibrium. While the magnitudes of the two multipliers are different, both deserve the adjective, 'large'. The initial percent drop in GDP in the linear and high-inflation ZLB equilibrium model is -2.18% and -2.84%,respectively. Again, while the numbers are different, the decline in output is large in both cases. In stark contrast, the properties of the low-inflation ZLB equilibrium are very different than those of the linear model. For example the impact multiplier is -0.35% and the initial drop in

sunspot inducing a switch form one equilibrium to the other. For example one could from the high inflation ZLB equilibrium associated with the initial level of G to the low inflation ZLB equilibrium associated with the high level of G. As in Mertens and Raven (2015), we abstract from this issue.

Table 3.1: Effect of r_t Shock

Model	Output	Inflation	Multiplier
Linear	-2.18	-0.0066	1.63
Nonlinear, high-inflation	-2.84	-0.0093	2.24
Nonlinear, low-inflation	-17.87	-0.0734	-0.35

GDP is 17.87%.

The size of the multiplier associated with the high-inflation ZLB equilibrium increases as p rises or θ falls, i.e. as the expected duration of the ZLB rises or as prices become more flexible. These results are consistent with the intuition in CER (2011) and EW. In contrast, the size of the multiplier associated with the low-inflation ZLB equilibrium become more negative as the multiplier increases as p rises or θ falls.

We found that increases in p and declines in θ have the effect of shifting the $f(\pi_\ell)$ function down. At some point it does not intersect zero at any of the values of π^ℓ considered. So the non-existence of an interior equilibrium in the non-linear models leads to an effective bounds on the multiplier. In practice we found that the upper and lower bounds associated with high and low inflation ZLB equilibria were 4.3 and -2.5 percent respectively.

The multiplier in the linear multiplier is inversely related

$$\Delta = (1-p)(1-\theta p) - p(2-\eta_g)\frac{(1-\theta)(1-\beta\theta)}{\theta}.$$

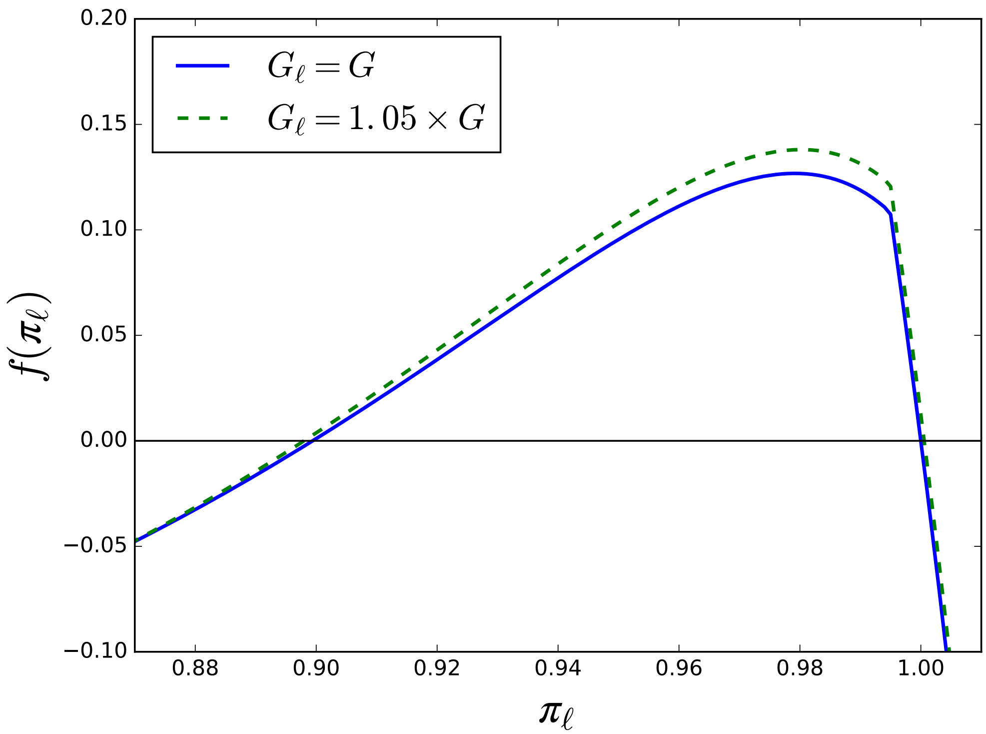
It is evident that the multiplier is strictly increasing in p and θ. See CER (2015) for the intuition underlying this result. Note that the high-inflation ZLB equilibrium of the non linear model is consistent with the comparative dynamics of the log linear model. The latter's implication for the effects of greater price flexibility and a longer lasting ZLB episode (in expectation) are qualitatively correct. Carlstrom, Fuerst and Paustian (2014) prove that the linear model does not have an interior equilibrium when Δ is negative. Before Δ turns negative, the multiplier can be *arbitrarily* large. So in contrast to the non-linear model, there is no upper bound to the multiplier.

To summarize, the basic qualitative results reported in CER using a log-linear approximation hold up when we consider the nonlinear solution and focus attention on the high-inflation ZLB equilibrium. In particular, (i) the government spending multiplier can be considerably bigger than unity when the ZLB binds, (ii) as the expected duration of the ZLB increases or the degree of flexibility of prices increases, then both the severity of the output collapse in the ZLB and the government spending multiplier are larger, (iii) for parameterizations in which the output collapse is large, then the government spending multiplier is large too. The implications of the linear approximation become increasingly distorted as parameter values are chosen for which Δ approaches zero and turns negative.

3.4. Sunspot Equilibria

Above, we assumed that a fundamental shock to preferences makes the ZLB bind. It is useful to also consider a scenario in which the ZLB binds because of a non-fundamental shock. This case is the one considered by Mertens and Ravn (2015). We assume that at the beginning of time $t = 0$, before any agent has made a decision, the economy is in the high-inflation steady state equilibrium. Each firms observes a sunspot. Conditional on the sunspot firms can either believe that other firms behave as in they did in the high inflation steady state or they set their prices sufficiently low to make the ZLB bind. With probability p firms continue to hold this belief. With

Figure 3.4: Steady State Sunspot Equilibrium Function

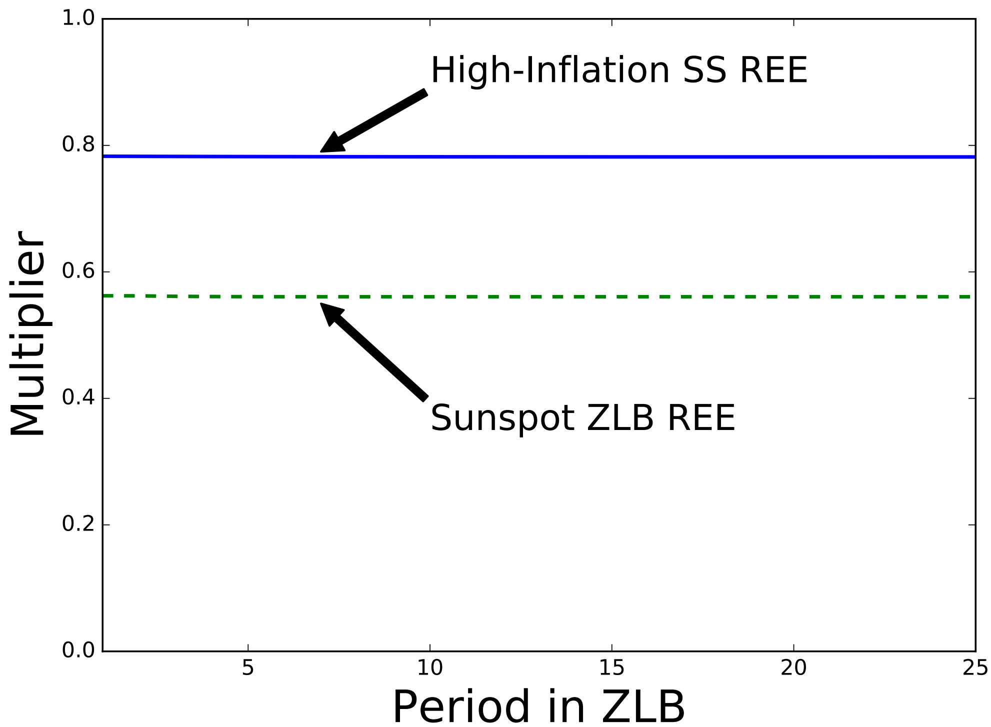


probability $(1-p)$, firms believe that other firms will set their prices sufficiently high to make the ZLB non-binding and behave as they did in high inflation steady state. The latter belief is an absorbing state.

Figure 3.4 displays the $f(\pi_\ell)$ function. Notice that there are two points at which it equals zero: the initial high inflation steady state and the low inflation ZLB equilibrium.

As stressed in Mertens and Ravn (2015), the sunspot equilibrium can be characterized as a situation in which the shock driving the economy into a binding ZLB is a loss in confidence. Agents anticipate deflation, creating the perception that the real interest rate is high. Households respond with a reduction in expenditures and thus

Figure 3.5: Sunspot Multipliers



drive the economy into a recession. The reduced level of economic activity results in a drop in marginal cost as the wage rate falls with the lower demand for labor. Reduced marginal costs create downward pressure on the price level. This pressure is manifested in the form of a sustained fall in the price level over time because of the presence of price-setting frictions. In this way, the initial fear of deflation is self-fulfilling. Mertens and Ravn (2015) propose this non-fundamental 'loss of confidence' shock as an alternative to a fundamental shock that drives the economy into the ZLB.

Now consider the effect of a rise in G when the sunspot occurs. Depending on beliefs, the economy will either in the high inflation steady state equilibrium or it could be in the low inflation ZLB equilibrium. We compute the multiplier assuming that the economy initially in the high inflation steady state and agents think the economy will go back there when the ZLB ends. Figure 3.5 displays the multiplier as a function of time for the case where the economy is in the ZLB and the case where the economy remains in the high inflation steady state equilibrium after the sunspot is operative. As might be anticipated, the multiplier at the ZLB can be larger or smaller than in the steady state, depending on parameter values. The robust result is that the multiplier is quite small: (0.56) in the ZLB and (0.79) at the steady state. The steady state multiplier is small because of the normal crowding of private consumption when the monetary induces a rise in the real rate in response to inflation.

4. Stability Under Learning at the ZLB

In this section we investigate the stability under learning of the high and low inflation ZLB equilibria. To determine what happens when agents don't know the precise equilibrium functions for the variables whose values they must forecast, we must make assumptions about how their beliefs evolve over time.

4.1. The benchmark case

Once we depart from the assumption of rational expectations, we must confront the following issue. In the Calvo model, intermediate good firms choose their price level, $P_{j,t}$, based in part on the value of the aggregate price level, P_t. But, the aggregate price level is a function of firms' collective price decisions. Clearly, these firms cannot actually 'know' the current aggregate price level when they choose their own price, in the sense of actually observing it. The standard assumption is that these firms form a 'belief' about P_t when they make their decision. In a rational expectations equilibrium that belief is correct. In a world where firms don't necessarily have rational expectations it is not natural to assume that firms actually see P_t when their collective actions simultaneously determine P_t. But if firms don't actually see P_t they also don't see C_t when they make their time t decisions.

So for firms to solve their problem they must have a view about the equilibrium functions for current and future aggregate inflation and consumption. At time t firms make their decisions given the state variable p^*_{t-1} and these views about the equilibrium functions. We assume that firms believe they are in a stationary environment so they think that the equilibrium functions won't change over time.

Denote by $x^{e,f}_{\ell}(p^*_{t-1}, t-1)$ the firms' belief, formed using information up to time $t-1$, about the equilibrium function for x_{ℓ}. The only argument of the function is the state variable p^*_{t-1}. While the firm knows the actual value of p^*_{t-1}, we attribute to it beliefs about the entire equilibrium function for x_{ℓ}. We do this for the following reasons. To make its time t decision, firms must forecast the values of future variables as p^*_t evolves. Put differently the firm's first order conditions involve objects like $x^{e,f}_{\ell}(p^*_{t-1}, t-1)$ and $x^{e,f}_{\ell}(p^*_{t+j}, t-1)$ for $j \geq 0$. It follows that the firms must have views about the entire function. The fact that all these functions have time $t-1$ as argument summarizes our assumption that firms think they are in a stationary environment.

Over time, firms observe data which cause their beliefs about the equilibrium functions to evolve. We assume that these beliefs evolve according to

$$x^{e,f}_{\ell}(p^*_t, t) = \omega x_{\ell}(p^*_{t-1}, t-1) + (1-\omega) x^{e,f}_{\ell}(p^*_{t-1}, t-1). \tag{4.1}$$

For $\omega > 0$, this formulation assumes that agents know the time $t-1$ equilibrium function for x_ℓ. While this assumption is clearly heroic, we also investigate what happens we see that firms just assume that the value of variables that they have to forecast are equal to their current value. As an aside, it is worth noting that in Rotemberg model, discussed in Section 6, there are no state variables in the ZLB. So we can replace (4.1) with the assumption that agents' expectations about the values of future variables evolve according to a simple constant gain algorithm.

When households make their time t consumption decisions, firms' actions have already determined aggregate inflation. Given this information, the households can compute the time t equilibrium function for inflation.[10] Denote by $\pi_\ell^{e,h}(p_{t-1}^*, t)$ households' belief, at time t, about the equilibrium function for π_ℓ. Absent any new information, households believe that this equilibrium function will be in effect forever. Given new information, households beliefs evolve according to

$$\pi_\ell^{e,h}(p_t^*, t+1) = \omega \pi_\ell(p_{t-1}^*, t) + (1-\omega)\pi_\ell^{e,h}(p_{t-1}^*, t). \tag{4.2}$$

The first-order condition of the firm when $r_t = r_\ell$ can be written as

$$\frac{\tilde{P}_{\ell,t}}{P_{t-1}} = \frac{P_t}{P_{t-1}} \frac{K_{\ell,t}^{e,f}}{F_{\ell,t}^{e,f}} \tag{4.3}$$

where

$$K_{\ell,t}^{e,f} = \chi \frac{(Y_{\ell,t}^{e,f})^2}{p_t^*} + \theta \frac{1}{1+r_\ell} (\pi_{\ell,t+1}^{e,f})^{\varepsilon} \left[p K_{\ell,t+1}^{e,f} + (1-p) K_{n,t+1}^{e,f} \right] \tag{4.4}$$

and

$$F_{\ell,t}^{e,f} = \frac{Y_{\ell,t}^{e,f}}{C_{\ell,t}^{e,f}} + \theta \frac{1}{1+r_\ell} (\pi_{\ell,t+1}^{e,f})^{\varepsilon-1} \left[p F_{\ell,t+1}^{e,f} + (1-p) F_{n,t+1}^{e,f} \right]. \tag{4.5}$$

[10] They can do so under the further heroic that they can just solve the problem that the firm solved.

Here, $F_{\ell,t}^{e,f}$, $K_{\ell,t}^{e,f}$, $Y_{\ell,t}^{e,f}$, $C_{\ell,t}^{e,f}$, and $\pi_{\ell,t}^{e,f}$ denote firms' beliefs about F_t, K_t, Y_t, C_t, and π_t when while $r_t = r_\ell$. Similarly, $F_{n,t}^{e,f}$, $K_{n,t}^{e,f}$, $Y_{n,t}^{e,f}$, $C_{n,t}^{e,f}$, and $\pi_{n,t}^{e,f}$ denote firms' beliefs about F_t, K_t, Y_t, C_t, and π_t when while $r_t = r$. Here, the superscript, 'e', indicates the firms' belief about the value of the corresponding variable. Note that there is a superscript, e, on the current period value of aggregate inflation and aggregate consumption and output. Firms form these beliefs based on their beliefs about equilibrium functions for aggregate variables, which evolve according to (4.1).

While $r_t = r_\ell$, households make their labor supply decision so that

$$\chi C_{\ell,t} h_{\ell,t} = \frac{W_{\ell,t}}{P_{\ell,t}} \tag{4.6}$$

They determine consumption so that

$$\frac{1}{C_{\ell,t}} = \frac{1}{1+r_\ell} \max\left\{1, 1+r+\alpha(\pi_{\ell,t}-1)\right\} \left[\frac{p}{C_{\ell,t+1}^{e,h}\pi_{\ell,t+1}^{e,h}} + \frac{1-p}{C_{n,t+1}^{e,h}\pi_{n,t+1}^{e,h}}\right] \tag{4.7}$$

where $C_{\ell,t}$, $h_{\ell,t}$, $R_{\ell,t}$, and $\frac{W_{\ell,t}}{P_{\ell,t}}$ are the realized values of consumption, labor supply, the nominal interest rate, and the real wage. Household beliefs about future inflation are determined by (4.2). Households believe that the function mapping the state p_{t-1}^* to the household consumption decision is the same in the subsequent period.

A learning ZLB equilibrium is a sequence of functions $\pi_{\ell,t}(\cdot)$, $C_{\ell,t}(\cdot)$, $h_{\ell,t}(\cdot)$, $\frac{W_{\ell,t}}{P_t}(\cdot)$, $\frac{\tilde{P}_t}{P_{t-1}}(\cdot)$, and $R_{\ell,t}(\cdot)$ that satisfy the resource constraint, the monetary policy rule, and the household and firm optimality conditions for all t, given an initial set of beliefs $\pi_\ell^{e,f}(\cdot,0)$, $C_\ell^{e,f}(\cdot,0)$, and $\pi_\ell^{e,h}(\cdot,0)$ that evolve according to (4.1) and (4.2). Note that we are taking the functions while $r_t = r$ as known by households and firms.

We define stability-under-learning in this economy as follows. Suppose that in a neighborhood of a ZLB steady state, either $x_\ell^{e,f}(\cdot, t-1)$ is not equal to $x_\ell(\cdot)$ or

$\pi_\ell^{e,h}(\cdot,t)$ is not equal to $\pi_\ell(\cdot)$. Here $\pi_\ell(\cdot)$ and $x_\ell(\cdot)$ are the rational expectations law of motion for x_ℓ and π_ℓ in a steady state ZLB equilibrium. A rational expectations ZLB equilibrium is said to be *stable-under learning* if a learning equilibrium with initial beliefs close to, but not equal to, the rational expectations equilibrium functions, converges back to the rational expectations ZLB equilibrium. If the economy stays in the ZLB forever, it will converge to the steady state ZLB equilibrium. The learning equilibrium must also approach that steady state ZLB REE if the initial ZLB REE is stable-under-learning. This fact is very useful because it allows us to eliminate all of the RE ZLB equilibria that lead to the low inflation ZLB steady state equilibrium as not being stable-under-learning.

Consider a firm that believes that the steady inflation rate is $\pi_\ell^{e,f}$ and that the economy is in the steady state corresponding to that rate of inflation. Also, assume that p_{t-1}^* is consistent with this belief. The belief $\pi_\ell^{e,f}$ is not a rational expectations belief so that the steady state associated with it (including p_{t-}^*) is not a steady state ZLB REE. It follows that $f(\pi_\ell^{e,f})$ is not equal to zero.

Note that there is an equivalence between the belief $\pi_\ell^{e,f}$ and the value of $\tilde{p}_\ell^{e,f}$ that will be chosen by firms who can update their price. So we use the function $f(\pi_\ell^{e,f})$ to define a new function

$$\tilde{f}\left(\tilde{p}_\ell^e\right) \tag{4.8}$$

that must be equal to zero at a steady state ZLB RE equilibrium.

Combining (4.3)-(4.5), using the aggregate resource (3.9) and the household Euler equation (4.7) we represent the first order condition of the firm under consideration as

$$\tilde{F}\left(\tilde{p}_t, \tilde{p}_\ell^{e,f}\right) = 0. \tag{4.9}$$

Define the best-response function

$$\tilde{p}_t = g(\tilde{p}_\ell^{e,f}).$$

This function has the property,

$$\tilde{F}\left(g(\tilde{p}_\ell^{e,f}), \tilde{p}_\ell^{e,f}\right) = 0. \tag{4.10}$$

Note that in a steady state RE ZLB equilibrium

$$\tilde{p}_t = \tilde{p}_\ell^{e,f} \tag{4.11}$$

If all firms behave like the firm under consideration, then time t inflation is given by

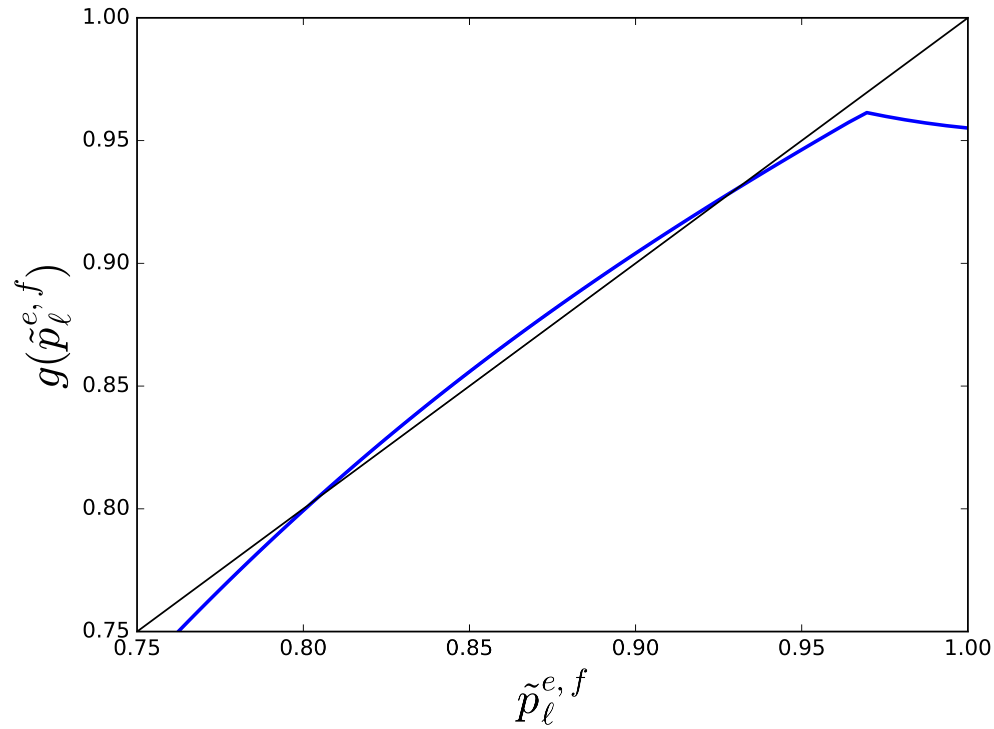
$$\pi_t = \left(\theta + (1-\theta)(\tilde{p}_t)^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}, \tag{4.12}$$

which will coincide with the typical firm's belief about the current aggregate inflation, given by

$$\pi_\ell^{e,f} = \left(\theta + (1-\theta)(\tilde{p}_\ell^{e,f})^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}. \tag{4.13}$$

Figure 4.1 plots the typical firm's best response function, i.e. $\tilde{p}_t$ as a function of $\tilde{p}_\ell^{e,f}$. The two steady state RE ZLB equilibria correspond to the two points where the best response function intersects the 45 degree line. Notice that given any belief, $\tilde{p}_\ell^{e,f}$, between the RE steady state beliefs, the best response $g(\tilde{p}_\ell^{e,f})$ is greater than $\tilde{p}_\ell^{e,f}$. It follows that realized inflation will exceed beliefs about inflation. So the learning equilibrium will move towards the high inflation ZLB REE steady state. Now consider any belief, $\tilde{p}_\ell^{e,f}$, that exceeds the high inflation RE steady state of inflation.

Figure 4.1: Best Response Function



Here the best response function $g(\tilde{p}_{\ell}^{e,f})$ is less than $\tilde{p}_{\ell}^{e,f}$. So realized inflation will be lower than beliefs about inflation and the learning equilibrium will move towards the high inflation RE steady state. Finally, consider any belief, $\tilde{p}_{\ell}^{e,f}$, that is less than the low inflation RE steady state of inflation. Here the best response function $g(\tilde{p}_{\ell}^{e,f})$ is less than $\tilde{p}_{\ell}^{e,f}$. It follows that realized inflation will be lower than beliefs about inflation. So the learning equilibrium will move away from the low inflation RE steady state.

The previous result discussion focused on the limiting point of RE ZLB equilibria. To be stable-under-learning, the functions defining a learning equilibrium must converge point wise to the functions defining a RE ZLB equilibrium for every possible for p_t^*, including the steady state value of p_{ℓ}^*. The previous discussion establishes that any RE ZLB equilibrium that converges to the low inflation RE ZLB steady state does not satisfy this condition, and is therefor not stable-under-learning.

It does not establish that a RE ZLB equilibrium that converges to the high inflation RE ZLB steady state does satisfy this condition and is therefore stable under learning. We must examine this issue numerically. Recall that we parameterize the RE ZLB equilibrium functions with a finite number of parameters, z_t. The learning algorithm specified above defines a mapping from the current values of those parameters to the values that they take in the subsequent period

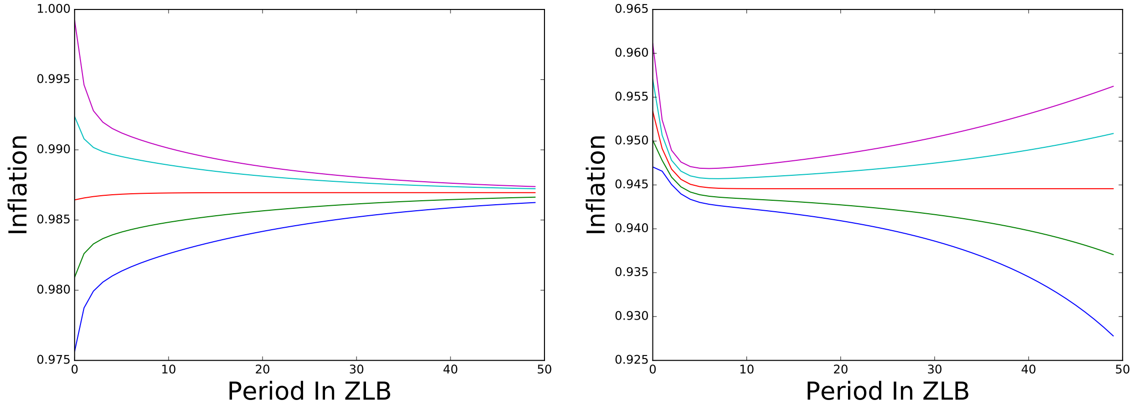
$$z_{t+1} = s\left(z_t\right). \tag{4.14}$$

Define

$$S(\tilde{z}) = \left[\frac{ds_i\left(z\right)}{dz_j}\right]|_{\tilde{z}}, \tag{4.15}$$

for all $i, j < N$ where N is the number of parameters. When we evaluate S for the

Figure 4.2: Learning Equilibria Near Steady State ZLB



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parameters of the the high inflation ZLB RE equilibrium, we find that the maximum eigenvalue is less than one in absolute value. This establishes that, locally, functions in the neighborhood of the high-inflation ZLB RE equilibrium will converge to the RE equilibrium functions in a learning equilibrium. By contrast, when we evaluate S for the parameters of the low-inflation ZLB RE equilibrium, we find that the maximum eigenvalue is greater than one in absolute value, meaning that functions in the neighborhood of the low-inflation ZLB RE equilibrium will diverge from the RE equilibrium functions in a learning equilibrium.

To illustrate the process of convergence and divergence, suppose that at time -1 the economy is in the high inflation ZLB steady state where $p^*_{-1} = p_\ell$. Then at time 0, for reasons unexplained, (i) $x^{e,f}_\ell(p^*_{-1}, -1) = x_\ell(p^*_{-1}) + \bar{x}_\ell$,where $\bar{x}_\ell$ is a positive constant, and (ii) all agents think that if the ZLB ends, the economy will be in a RE equilibrium that converges to the high inflation steady state.[11] For simplicity we assume that the parameter ω in (4.1) and (4.2) is equal to one.

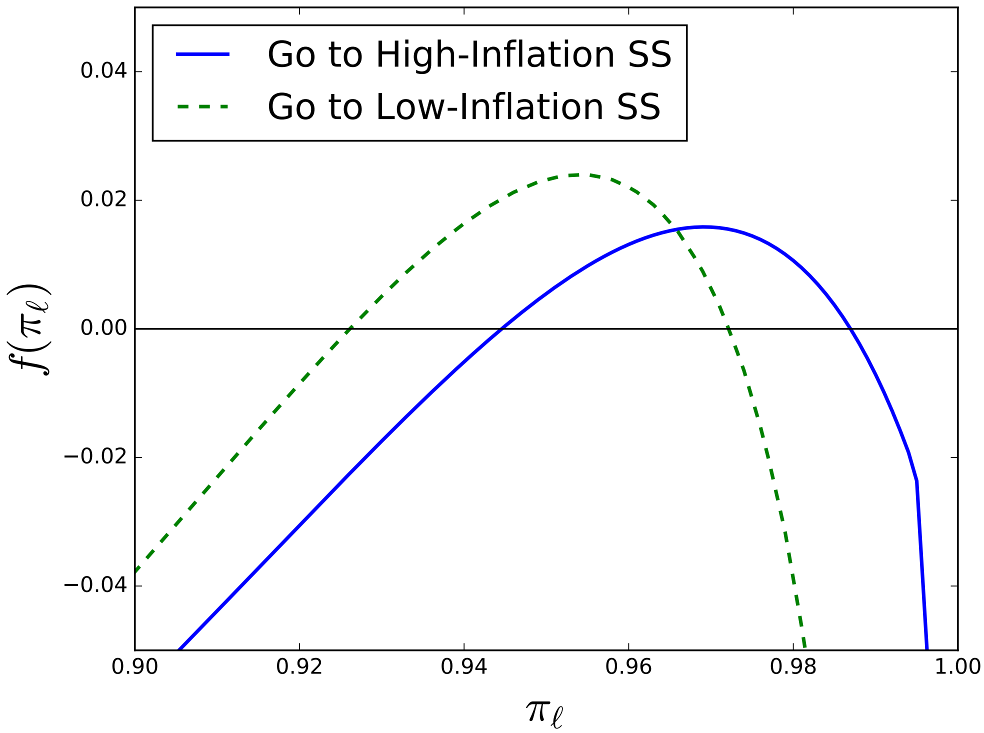
[11] We obtain virtually identical results regardless of whether $\bar{x}_\ell$ is applied to firms' beliefs about only inflation, only consumption or both.

The first panel of Figure 4.2 displays the evolution of realized inflation for $\bar{x}_\ell$= (-.02,-.01, 0, 0.01, 0.02). The red line corresponding to $\bar{x}_\ell = 0.0$ is the inflation rate in the high inflation steady state RE ZLB equilibrium. From the Figure we see that, regardless of the value of $\bar{x}_\ell$, inflation converges to the high inflation steady state ZLB equilibrium. This result established that the equilibrium defining a learning equilibrium converges to equilibrium function defining an RE equilibrium when evaluated at the p^*_ℓ. The second panel is the analog to the first, where we begin from the low inflation steady state RE ZLB equilibrium. Notice that inflation diverges from that equilibrium in the learning equilibrium. For positive values of $\bar{x}_\ell$, inflation converges to the high inflation steady state ZLB RE equilibrium. Interestingly for $\bar{x}_\ell < 0$, there does not exist interior ZLB learning equilibrium.

Until now we supposed that agents belief that once the ZLB is over, the economy will go to an RE equilibrium that converges to the high inflation steady state. It is natural to ask what happens if agents assume that when the ZLB is over, the economy will go to an RE equilibrium that converges to the low inflation steady state. Figure 4.3 is the analog to Figure 3.1 for this alternative assumption. Notice that the curve is shifted to the left, meaning that there are two steady state ZLB equilibria, and their inflation rates are lower than under our standard assumptions. The reason the curve is shifted to the left is that agents expect a lower rate of inflation after the ZLB is over. This effect means that the real interest in the ZLB is higher which lowers consumption.

It is still the case that RE equilibria which converge to the low inflation steady state are not stable-under-learning. It is also still the case that the maximum eigenvalue associated with the matrix in (4.15) is less than one in absolute value. So regardless of which assumption we make about agents beliefs about the post ZLB period the high inflation ZLB RE equilibrium is stable-under-learning and the low

Figure 4.3: Steady State ZLB Equilibrium Function, Alternative SS Expectations



inflation ZLB RE equilibrium is not.

We conclude by noting, that there may be multiple RE ZLB equilibria that converge to the high inflation steady state ZLB equilibrium. But as a practical matter we could find not any of those equilibria. As it turns out, this potential ambiguity does is resolved once we consider return to the Rotemberg model.

5. Fiscal Policy in the Learning Equilibrium

In the previous section we discussed the government multiplier in RE ZLB equilibria that converge to the high and low inflation steady states. The effect of learning dynamics and fiscal policy is interesting because the Great Recession was such an unusual event. Under such circumstances, it may questionable to assume that people had rational expectations about all aspects of the episode.

5.1. Fiscal policy under benchmark learning scheme

In what follows we analyze the value of government spending multipliers in the learning equilibrium. We initially assume that agents think that when the ZLB episode is over, the economy reverts to the high inflation SS. Later we assess the robustness of our results to this assumption.

We imagine that the economy begins in the high inflation steady. At time 0, $p^*_{-1} = 1$, r falls to r_ℓ and obeys the law of motion above. Agents initially believe that equilibrium functions correspond their rational expectations, where agents thought r_t would be r forever. Firms' and households beliefs about equilibrium functions evolve according to (4.1) and (4.2).

Figure 5.1 displays the paths of consumption, inflation, and the government spending multiplier in the learning equilibrium (the blue lines), as well as the high-

Figure 5.1: Learning Equilibrium, Starting from Steady State

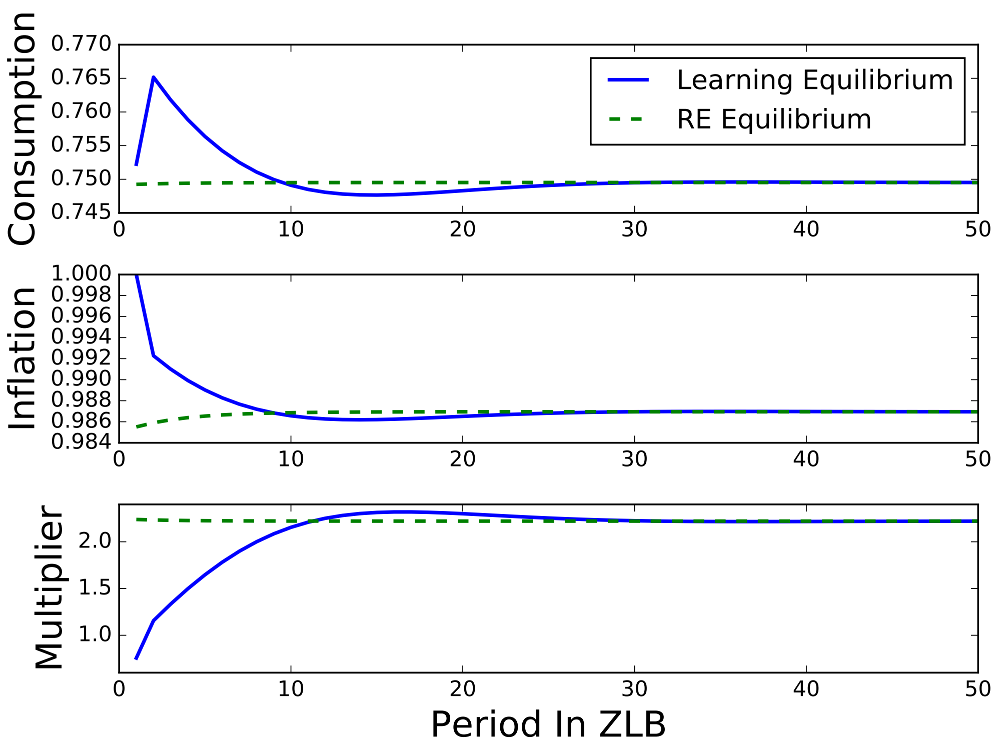
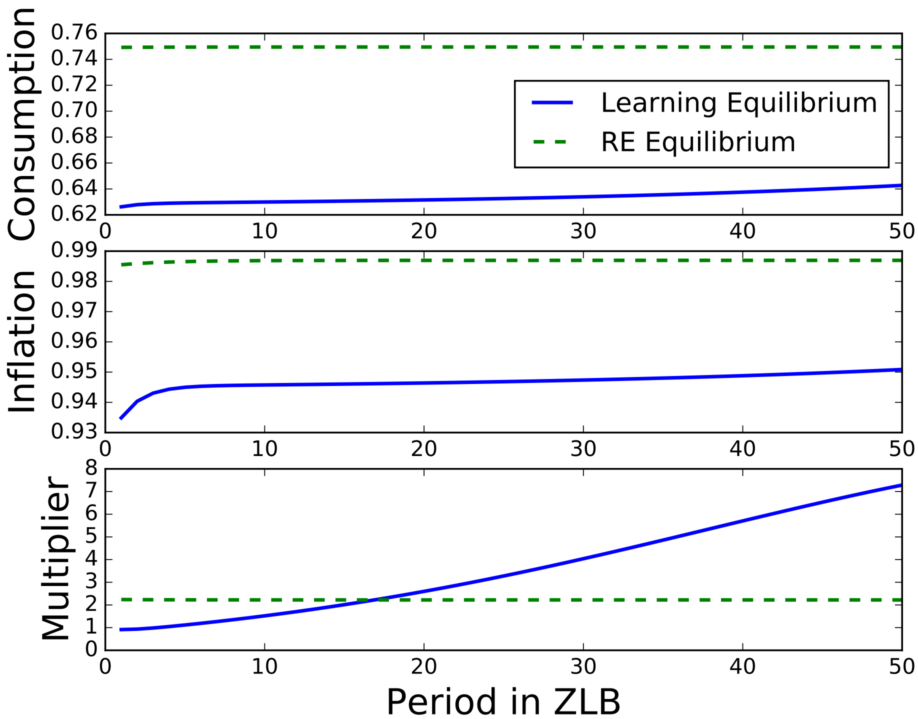


Figure 5.2: Learning Equilibrium, Starting Near Low-Inflation RE ZLB Equilibrium



inflation RE ZLB paths (the green lines). The paths for inflation and consumption are computed holding government consumption at its steady state value (0.20). Notice that consumption and inflation converge to high inflation RE ZLB equilibrium from above. The reason that they initially take on higher values is that expectations about higher future inflation and consumption spur demand in the present. As expectations adjust downward with realized inflation and consumption, they push inflation and consumption down further. The multiplier starts out low because the ZLB is not binding in the first few periods. Once the ZLB starts to bind, the multiplier quickly rises above 1.

An alternative experiment is to imagine that after the shock to r_t, firms and household have beliefs near the low-inflation RE ZLB equilibrium. Figure 5.2 displays the paths of consumption, inflation, and the government spending multiplier in the learning equilibrium (the blue lines), as well as the high-inflation RE ZLB paths (the green lines). As before, the paths for inflation and consumption are computed holding government consumption at its steady state value (0.20). Notice that consumption and inflation converge to high inflation RE ZLB equilibrium from below. The reason that they initially take on lower values is that expectations about low future inflation and consumption depress demand in the present. The multiplier starts out around 1 and then rises after that. The reason the multiplier rises is that the fiscal expansion helps quickly move expectations toward the high-inflation RE ZLB equilibrium expectations. Without the change in government spending, expectations remain close to the low-inflation RE ZLB equilibrium for some time. Notably, the multiplier continues to rise for some time. After many periods, the multiplier eventually approaches the high-inflation RE ZLB steady-state equilibrium value.

5.2. Sensitivity Analysis

Of course, the learning setup that we have specified implies that while $r_t = r_\ell$ households and firms learn about the mapping from p^*_{t-1} to equilibrium prices and quantities. This ascribes to them a large amount of sophistication. Perhaps it is more reasonable to assume that households and firms expect that tomorrow's inflation rate and consumption will be equal to the observed value today, so long as r_t remains low. In this simple learning equilibrium setup, households and firms ignore the state variable p^*_{t-1} while $r_t = r_\ell$ and form expectations is a way that is consistent with rational expectations only at the steady state RE ZLB equilibrium point.

In the previous section we assumed that agents update their beliefs about equilibrium *functions.* Here, we assess the robustness of our results to assuming that agents update about their beliefs about the *values* of variables that they must forecast. In particular we assume that firms' beliefs evolve according to the constant-gain-rule

$$Z^{f,\ell}_{t+j|t} = \omega Z_{t-1} + (1-\omega) Z^{f,\ell}_{t|t-1} \text{ for all j} \geq 0. \tag{5.1}$$

Here $Z^{f,\ell}_{t+j|t}$ denotes the expectations that the firm has about the values at time $t+j$ of consumption and inflation based on their time t information set. The superscript ℓ denotes that agents are forecasting the value of variables in the ZLB. In the special case of $\omega = 1$, firms just assume that variables are martingales. This assumption is correct in the rational expectations steady state ZLB equilibrium.

We assume that household expectations about the time $t+j$ value of inflation evolves according to

$$\pi^{h,l}_{t+j|t} = \omega \pi_t + (1-\omega) \pi^{h,l}_{t|t-1} \text{ for all j} \geq 1. \tag{5.2}$$

Analogous to Figure 5.1, in Figure 5.3 we assume that households and firms think that consumption and inflation will remain at their steady state values in the period that r_t falls to r_ℓ. The blue lines plot the learning equilibrium holding g at its steady state value and the green dashed lines plot the learning equilibrium with g equal to 1.05 times its steady state value. Even under our simple learning setup that ignores the state variable p^*_{t-1}, consumption and inflation quickly approach their high-inflation steady state RE ZLB values. Moreover, the multiplier rises above 1 as soon as the ZLB begins to bind and then approaches the steady state RE ZLB equilibrium value.

Analogous to Figure 5.2, in Figure 5.4 we assume that households and firms think

Figure 5.3: Simple Learning Equilibrium, Starting With Expectations at SS

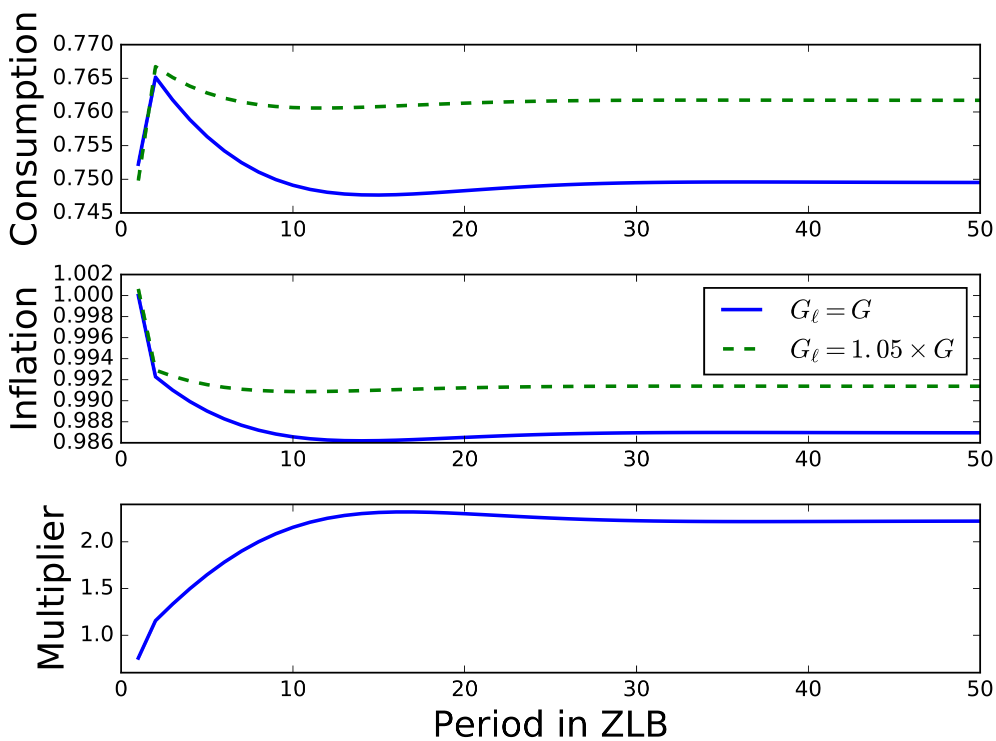
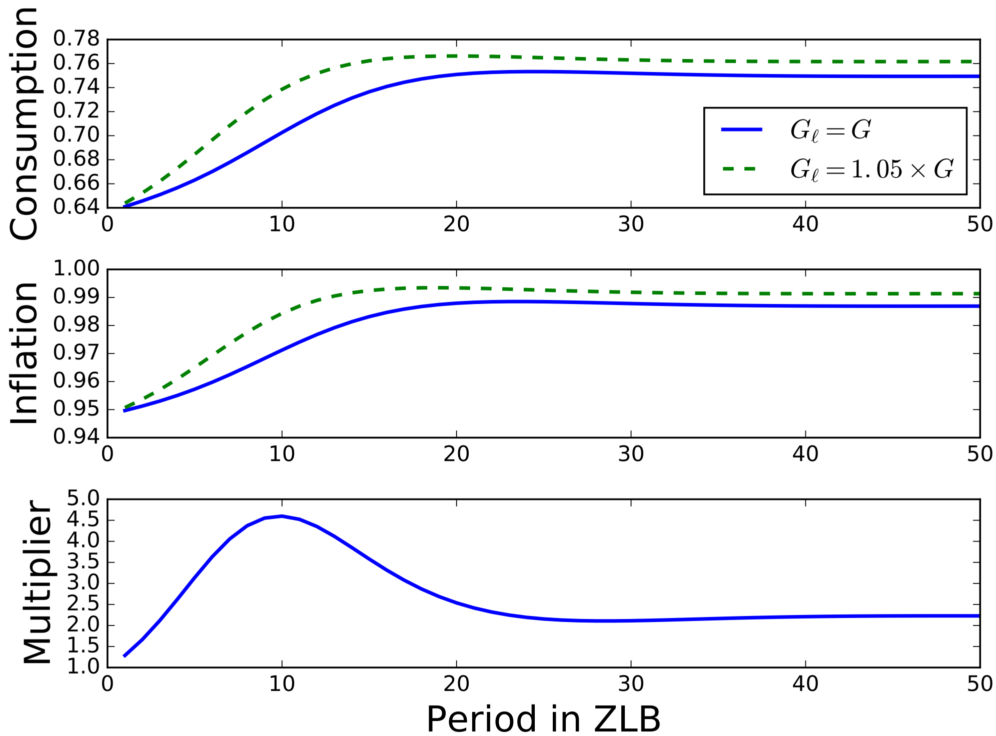


Figure 5.4: Simple Learning Equilibrium, Starting With Expectations near Low Inflation SS RE ZLB



that consumption and inflation will be near the low-inflation steady state RE ZLB equilibrium in the period that r_t falls to r_ℓ. As in the previous figure, the blue lines plot the learning equilibrium holding g at its steady state value and the green dashed lines plot the learning equilibrium with g equal to 1.05 times its steady state value. Again, consumption and inflation approach their high-inflation steady state RE ZLB values. Moreover, the multiplier begins above 1 and rises for some time thereafter before approaching the steady state RE ZLB equilibrium value from above.

In both experiments, ignoring the state variable causes the speed of convergence to the high-inflation RE ZLB equilibrium to increase dramatically. The robust prediction from our learning experiments is that while $r_t = r_\ell$, the paths for consumption and inflation converge to the high-inflation steady state ZLB REE value, and the multiplier is large along this path, so long as they do not start too far away. We have never found paths that converge to the low-inflation steady state ZLB REE.

Finally, in the appendix we show that none of our results regarding stability are affected if we have households and firms learn as in Evans and Honkapohja (2001) or which values of $\omega > 0$ that we use. What is affected is the speed of converge to a stable equilibrium and speed of divergence from an unstable equilibrium. In sum, our sensitivity analysis corroborates our basic finding that the high-inflation ZLB REE is stable under learning. The analog low-inflation equilibrium is not and is therefore not empirically interesting.

5.3. Reconciling with Mertens and Ravn (2015)

Mertens and Ravn (2015) report that the fiscal multiplier is small when they analyze a learning equilibrium near the low inflation steady state RE ZLB equilibrium. This result contrasts sharply with our result that the multiplier is very large when start near the same ZLB equilibrium. There are four differences our analysis and theirs'. First, they work with a linearized Calvo model when they study the learning equilibrium. Second, they assume that firms who choose prices at time t, see the time t aggregate price level when they choose prices. Third, Mertens and Ravn suppose that households and firms learn as in Evans and Honkapohja (2001). In contrast we suppose that households believe that the function mapping the state p^*_{t-1} to the household consumption decision is the same in the subsequent period. Fourth, the experiment that underlies their multiplier calculation is subtly but very significantly than ours. When we calculate the multiplier we initially consider an economy in which agents initial expectations about inflation differ by ϵ_π from the low inflation steady state ZLB REE. We then consider a separate economy with shocks that set r to r_ℓ and a shock to G. Expectations start in the same place for the two economies. We then use the difference in output between the two economies to calculate the multiplier. Mertens and Ravn (2015) proceed in the same way with one crucial difference that is best explained as follows. When G increases, the rate of inflation in the ZLB REE falls by ϵ'_π. When the Mertens and Ravn raise G in the learning equilibrium, they also decrease agents' expectations about inflation by ϵ'_π. Notice that as discussed above, in and of itself this fall in inflation reduces output in the ZLB.

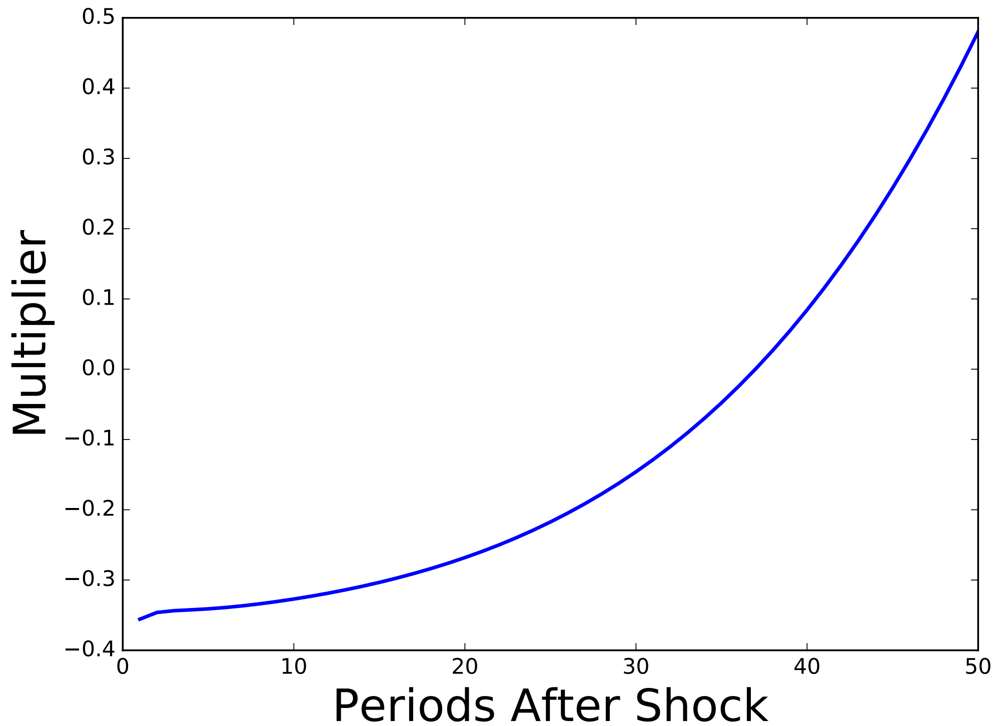
In the Appendix we show that first three differences between our analysis and Mertens and Ravn (2015) do not play a large role in the fully non-linear Calvo model with our benchmark learning scheme.[12] In contrast the fourth difference is very important. Figure 5.5 displays what the multiplier as a function of time if we adopt the assumption of Mertens and Ravn (2015) about how expectations about inflation change when G increases. Notice that we obtain a *negative* multiplier that persists for roughly 10 years. This results reflects that, in this example, the change in expectations is quantitatively much more important than the increase in G. Our own view is that their experiment confounds two shocks in the learning equilibrium.

6. The Rotemberg Model

A number of authors have studied the behavior of the economy in the ZLB interpreting the price frictions in the EW analysis as stemming from adjustment costs as proposed by Rotemberg (1982). A prominent example in this literature is Braun, Boneva and Waki (2015) who study the accuracy of linear approximations to the

[12] With the simple learning scheme, adopting Mertens and Ravn assumption about households (the third difference) results in a slower rate of convergence to the high inflation steady state ZLB equilibrium.

Figure 5.5: Multiplier with Shock to Expectations



model.[13] This interpretation of the Calvo model is interesting because it implies the same linearized equations that EW study. An alternative approach which also implies the linearized equations studied by EW is based on the price setting frictions proposed by Calvo. The advantage of adopting Rotemberg adjustment costs here is analytic simplicity. The Calvo approach injects an endogenous state variable (past price dispersion), while there is no endogenous state variable in the Rotemberg approach. In this section we highlight an important potential shortcoming of using Rotemberg adjustment costs when studying multiplicity and learnability issues.

With once exception, the Rotemberg model is identical to the Calvo model discussed above. The exception is that instead of (3.6) - (3.7) we assume that the monopolist who produces the j^{th} good has the following objective:

$$E_t \sum_{k=0}^{\infty} \beta^k \lambda_{t+k}[(1+\nu)\frac{P_{j,t+k}}{P_{t+k}} Y_{j,t+k} - s_{t+k} Y_{j,t+k} - \Phi_{t+k}\left(\frac{P_{j,t+k}}{P_{j,t+k-1}} - 1\right)^2]. \tag{6.1}$$

The variable Φ_t denotes a potentially state dependent function that scales the firm's costs of adjusting prices. In the classic Rotemberg model,

$$\Phi_t = \phi \tag{6.2}$$

To accommodate growth, Christiano and Eichenbaum (2012) assume

$$\Phi_t = \frac{\phi}{2}(C_t + G_t). \tag{6.3}$$

In contrast, authors like Braun et. al. (2015) and Gust, Herbst, Lopez-Salido and

[13] Braun, Boneva, and Waki (2015) paper was first written in 2012. As best as we can tell, it is the first paper to analyze the accuracy of the linearized EW model of the ZLB relative to the underlying nonlinear model.

Smith (2015), assume

$$\Phi_t = \frac{\phi}{2} Y_t. \tag{6.4}$$

As it turns out, existence and learnability of equilibria in the Rotemberg model depend on exactly which specification of Φ_t one adopt.

In the Appendix we show that an interior minimum state variable equilibrium for all three versions of the Rotemberg model is a set of eight *numbers*:

$$\pi, C, R, h, \pi_\ell, C_\ell, R_\ell, h_\ell,$$

that, when $r_t = r_\ell$, satisfy:

$$R_\ell = \max\left\{1, \frac{1}{\beta} + \alpha\left(\pi_\ell - 1\right)\right\} \tag{6.5}$$

$$\frac{1}{R_\ell} = \frac{1}{1+r_\ell}\left[p\frac{C_\ell}{\pi_\ell C_\ell} + (1-p)\frac{C_\ell}{\pi C}\right] \tag{6.6}$$

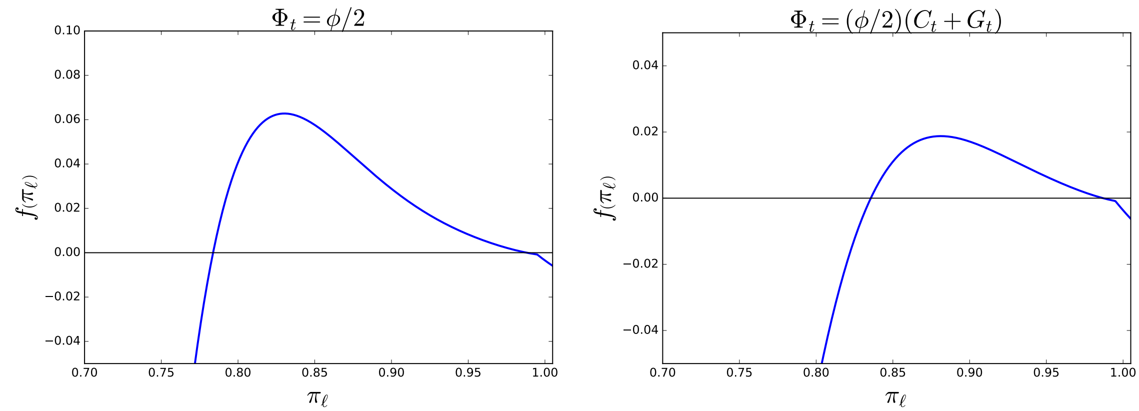
$$h_\ell = C_\ell + G_\ell + \Phi_\ell\left(\pi_\ell - 1\right)^2 \tag{6.7}$$

$$\begin{aligned}\left(\pi_\ell - 1\right)\pi_\ell = {} & \frac{1}{2\Phi_\ell}\varepsilon\left(\chi h_\ell C_\ell - 1\right)\left[C_\ell + G_\ell + \Phi_\ell\left(\pi_\ell - 1\right)^2\right] \\ & + \frac{1}{1+r_\ell}\left[p\left(\pi_\ell - 1\right)\pi_\ell + (1-p)\left(\pi - 1\right)\pi\frac{C_\ell}{C}\frac{\Phi}{\Phi_\ell}\right]\end{aligned} \tag{6.8}$$

Subscript ℓ denotes the value of a variable when $r_t = r_\ell$ and no subscript denotes the value of a variable after $r_t = r$.

A key difference between this model and the Calvo is the absence of any state variable in the ZLB. We solve these equations as follows. Conjecture a value for π_ℓ. Use (6.5) to solve for R_ℓ. Since we know the values of all variables outside the ZLB (variables with no subscript), we use (6.6) to determine C_ℓ. Equation (6.7) the

Figure 6.1: $f(\pi_\ell)$ in Rotemberg Model



determines h_ℓ. We then check whether (6.8) holds. If it does, we have a ZLB REE. If not we choose a different value for π_ℓ.

The equations defining a RE equilibrium collapse into one equation in one unknown, π_ℓ,

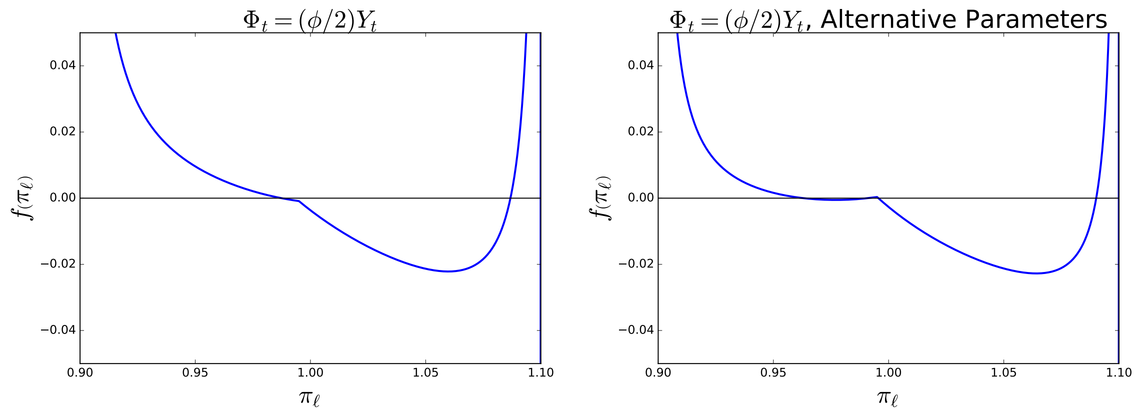
$$f(\pi_\ell) = 0. \tag{6.9}$$

This equation is analogous (3.12) in the Calvo model. The key difference is that the latter is an equation that determines the steady state ZLB REE. Since there is no state variable in the Rotemberg model, (6.9) determines the ZLB equilibrium values so long as $r_t = r_\ell$.

The two panels of Figure 6.1 plot $f(\pi_\ell)$ for Φ_t given by (6.2) and (6.3). In all cases we use the benchmark parameters given in (3.13). The parameter ϕ is chosen so that the log-linearized implies the same system of equations implied by the log-linearized Calvo mode, respectively.[14] The domain of admissible values of π_ℓ is restricted by

[14] Given our normalization that steady state output is one, this requirement implies that ϕ satisfies $(\varepsilon - 1)\,\phi = \frac{(1-\theta)(1-\beta\theta)}{\theta}$.

Figure 6.2: $f(\pi_\ell)$ in Rotemberg Model



the conditions that $C_\ell > 0$ and $Y_\ell > 0$. In the Appendix we detail the different domains for the different specifications of Φ_t.

Two features of the figures are worth noting. First, the plots of $f(\pi_\ell)$ are very similar when Φ_t is given by (6.2) or (6.3). Second, there are two ZLB equilibria, both of which feature deflation. Note that the curve looks very similar to the analogous curve that determines the two steady state ZLB REE in the non-linear Calvo model.

The two panels of Figure 6.2 display, for different parameter values, $f(\pi_\ell)$ for Φ_t given by (6.4). The first panel pertains to our benchmark values. Notice that the plot of $f(\pi_\ell)$ looks very different than the cases when Φ_t is given by (6.2) or (6.3). There is in fact a unique ZLB equilibrium which has an extremely similar level of inflation to the relatively low deflation ZLB equilibria displayed in Figure 6.1. Interestingly there is also a unique non-ZLB equilibrium when $r_t = r_\ell$ that is associated with a substantial amount of inflation.

The second of Figure 6.2 is the analog to the first except that we consider different values of the model's parameters:

$$\begin{aligned} \varepsilon &= 7.0,\ \beta = 0.99,\ \alpha = 2.0,\ p = 0.83, \\ r^{\ell} &= -0.0001,\ \phi = 200,\ \eta_g = 0.2, g_{\ell} = 0.23 \end{aligned}$$

Strikingly there are now two ZLB equilibria and two non-ZLB equilibria when $r_t = r_\ell$. This example is consistent with results in Braun et. al. (2015). A striking feature of both panels in Figure 6.2 is that $f(\pi_\ell)$ has two asymptotes, at quarterly rates of deflation and inflation of 10%. At these rates of inflation, the costs of adjustment consume all of output so that consumption can no longer be non-negative.

It is easy to characterize which ZLB equilibria are stable under learning for the Rotemberg model. Going from left to right in the plots, whenever $f(\pi_\ell)$ crosses from above, the equilibrium is stable under learning. From figure 6.1, when Φ_t is given by (6.2) or (6.3) there is a unique equilibrium that's is stable under learning. That equilibrium is the one with less deflation. When Φ_t is given by (6.4) and we work with the benchmark parameter values there is only one ZLB equilibrium. Notably, the non-ZLB equilibrium is not stable when adjustment costs are given by (6.4). Even with two ZLB equilibria, as in the second panel of 6.2, there is only one that's stable under learning. Interestingly, that equilibrium is the one that has more deflation. However, we are unable to find any similar situation when adjustment costs are scaled by (6.2) or (6.3), or in the Calvo model.

Unlike the Calvo model, the multiplier in the ZLB for the Rotemberg model is constant. Table 6.1 summarizes the values of the multiplier for the equilibria in Figures 6.1 and 6.2 that are stable. The multipliers in the stable equilibria are remarkably similar and to the multiplier in the linear Calvo model (1.63). Recall that the impact value of the ZLB multiplier in the unique learnable non linear Calvo

Table 6.1: Multipliers in the Rotemberg Model

Adj. Cost	Stable Equilibrium	Unstable Equilibrium
$\Phi_t = \frac{\phi}{2}$	1.56	0.98
$\Phi_t = \frac{\phi}{2}(C_t + G_t)$	1.70	0.36
$\Phi_t = \frac{\phi}{2} Y_t$	1.65	1.07

model was 2.24. Viewed our results strongly support the view that once we focus on learnable equilibria, the implications of the NK model for multipliers in the ZLB are very robust: the multiplier is large and increasing the more binding is the ZLB.

When adjustment costs are scaled by (6.2) or (6.3), we are able to find sunspot equilibria similar to the equilibria studied by Mertens and Ravn (2015). However, when adjustment costs are scaled by (6.4), there is no such ZLB equilibrium under our baseline parameterization. Instead, the sunspot equilibrium exhibits high inflation. Again, we find that the sunspot equilibrium is not stable under learning.

Overall, we conclude that the scaling term of price-adjustment costs in the Rotemberg model can have a large effect on the properties of the equilibria that we find that are not stable under learning. This is true even though the models have identical linearizations. If adjustment costs are scaled by (6.2) or (6.3), then the equilibria we studied in the Calvo model also exist in the Rotemberg model, and they have similar properties. Under our baseline parameterization, for all of the specifications of adjustment costs that we consider in the Rotemberg model, there always exists an equilibrium that has properties similar to the equilibrium in the Calvo model that is stable under learning.

7. Conclusion

In this paper we analyze whether the non-uniqueness of equilibria in NK models poses a substantive challenge to the key conclusions in the literature about the efficacy of fiscal policy in ZLB episodes. We argue that it does not. This conclusion rests on our view that if an REE is not stable-under learning, then it is simply too fragile to be taken seriously as a description of the data. We make our argument using particular models of learning. While we have explored alternative learning mechanisms, it is certainly possible that there exist alternative learning models for which our results do not go through. Still we believe our results are very supportive of the view that the key properties of linearized NK models regarding the impact of changes in government consumption in the ZLB are robust and should be taken seriously.

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